Eventually Colorings of closed maps of metric spaces

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イタリア/ガゼルタ June 2012

Abstract. For each natural number p we define eventual colorings of maps which are generalized notions of colorings of fixed-point free maps of metric spaces. We investigate the eventual coloring number $C(f,p)$ of a fixed-point free closed maps $f : X \to X$ with zero-dimensional set of periodic points. In particular, we show that if $\mathrm{ord}(f) = \sup\{|f^{-1}(x)|\ x \in X\} < \infty$ and $\dim X < \infty$, then there is a natural number p, which depends on $\dim X$, and X can be devided into two closed regions C_1 and C_2 such that for each point $x \in X$, the orbit $\{f^k(x)\}_{k=0}^{\infty}$ of x goes back and forth between C_1 and C_2 within time p.

Let $f : X \to X$ be a fixed-point free closed map of a metric space X, i.e., $f(x) \neq x$ for each $x \in X$. A subset C of X is called a *color* of f if $f(C) \cap C = \phi$. Note that $f(C) \cap C = \phi$ if and only if $C \cap f^{-1}(C) = \phi$. We say that a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is called the *coloring number* of f. The coloring number $C(f)$ has been investigated by many mathematicians.

Theorem 1. (Lusternik and Schnirelman) *Let* $f : S^n \to S^n$ *be the antipodal map of the* n*-dimensional sphere* S^n*. Then* $C(f) = n + 2$.

Theorem 2. (Aarts, Fokkink and Vermeer) *Let* $f : X \to X$ *be a fixed-point free involution of a metric space* X *with* $\dim X = n < \infty$*. Then* $C(f) \leq n + 2$.

Theorem 3. (Aarts, Fokkink and Vermeer) *Let* $f : X \to X$ *be a fixed-point free homeomorphism of a metric space* X *with* $\dim X = n < \infty$*. Then* $C(f) \leq n + 3$.

Let $f : X \to X$ be a fixed-point free map of a space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within* p of f if

$$\bigcap_{i=0}^{p} f^{-i}(C) = \phi.$$

Note that C is a color of f if and only if C is eventually colored within 1.

Proposition 4. *Let $f : X \to X$ be a fixed-point free map of a metric space X and $p \in \mathbb{N}$. Then a subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e., for each $x \in C$, $f^i(x) \notin C$ with some $i \leq p$.*

We define the eventual coloring number $C(f,p)$ as follows. A cover $\mathcal{C}$ of X is called an *eventual coloring within* p if each element C of $\mathcal{C}$ is eventually colored within p. The minimal cardinality $C(f,p)$ of all closed (or open) eventual colorings within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. If there is some $p \in \mathbb{N}$ with $C(f,p) < \infty$, we say that f is eventually colored.

For evaluating the eventual coloring numbers of maps, we define the following index $\psi_n(k)$. Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n+1$. Put $R(n,k) = n - (n+2-k)[\frac{n}{n+2-k}]$, where $[x] = \max\{m \in \mathbb{N} \cup \{0\} | \ m \leq x\}$ for $x \in [0, \infty)$. Note that $R(n,k)$ means the remainder of n divided by $(n+2-k)$. First, we put $\psi_n(0) = 1$ $(k = 0)$. Next we consider the following two cases (i) and (ii):

$$\text{(i) } R(n,k) < n+1-k.$$

$$\text{(ii) } R(n,k) = n+1-k.$$

For each $1 \leq k \leq n+1$, we define the index $\psi_n(k)$ by

$$\psi_n(k) = \begin{cases} k(2[\frac{n}{n+2-k}] - 1) + 2 & (\text{if } R(n,k) < n+1-k), \\ k(2[\frac{n}{n+2-k}] + 1) + 1 & (\text{if } R(n,k) = n+1-k). \end{cases}$$

For any map $f : X \to X$, $P(f)$ denotes the set of all periodic points of f. The following result is the main theorem of my talk.

Theorem 5. *Let $f : X \to X$ be a fixed-point free closed map of a metric space X with ord*$(f) = \sup\{|f^{-1}(x)|\ x \in X\} < \infty$ *and* $\dim X = n < \infty$. *If* $\dim P(f) \leq 0$, *then*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n+1$. In particular, if f is a homeomorphism, then

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n+1$.

For the compact case, we do not need the condition "ord$(f) < \infty$". In fact, we have the following.

Theorem 6. *Let $f : X \to X$ be any fixed-point free map of a compact metric space X with* $\dim X = n < \infty$. *If* $\dim P(f) \leq 0$, *then*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Corollary 7. *Let* $f : X \to X$ *be a fixed-point free closed map of a metric space* X *such that ord*$(f) < \infty$, $\dim X = n < \infty$ *and* $\dim P(f) \leq 0$. *Then there is some* p *such that* $C(f,p) = 2$.

In other words, X *can be divided into two closed subsets* C_1, C_2 *(i.e.,* $X = C_1 \cup C_2$*) and there is some* $p \in \mathbb{N}$ *such that if* $x \in C_i$ $(i \in \{1,2\})$, *there is a strictly increasing sequence* $\{n_x(k)\}_{k=1}^{\infty}$ *of natural numbers such that* $1 \leq n_x(1) \leq p$, $n_x(k+1) - n_x(k) \leq p$ *and if* $j \in \{1,2\}$ *with* $j \neq i$, *then*

$$f^{n_x(k)}(x) \in C_j - C_i \text{ (k:odd)}, \ f^{n_x(k)}(x) \in C_i - C_j \text{ (k:even)}.$$

Corollary 8. *Let $f : X \to X$ be a fixed-point free closed map of a metric space X such that ord*$(f) < \infty$, $\dim X = n < \infty$ *and* $\dim P(f) \leq 0$. *Then the followings hold.*

(1) If $\dim X = 0$, *then* $C(f, 2) = 2$.

(2) If $\dim X = 1$, *then* $C(f, 7) = 2$.

(3) If $\dim X = 2$, *then* $C(f, 16) = 2$.

(4) If $\dim X = 3$, *then* $C(f, 29) = 2$.

(5) If $\dim X = 4$, *then* $C(f, 46) = 2$.

(6) If $\dim X = 5$, *then* $C(f, 67) = 2$.

Remark. If we do not assume $\dim P(f) \leq 0$, the above theorems are not true. Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Note that $P(f) = S^n$ and $C(f,p) = C(f,1) = n+2$ for any $p \in \mathbb{N}$.

In the above resulst for noncompact case, we need the condition; $\mathrm{ord}(f) < \infty$.

Example. There are a (zero-dimensional) separable metric space X and a fixed-point free map $f : X \to X$ such that $\dim P(f) \leq 0$ and
(1) f is closed,
(2) f is finite-to-one, and
(3) f cannot be eventually colored within any $p \in \mathbb{N}$.
In fact, let $f : X \to X$ be the map as in [7, Theorem 3.12.7]. Then we see that $P(f) = \phi$ and f cannot be colored and satisfies the conditions (1), (2) (see [7, Theorem 3.12.7]). Let $\mathcal{U}$ be any finite open cover of X. Then there exist some $U \in \mathcal{U}$ and a point $x \in U$ such that $f^p(x) \in U$ for any $p \in \mathbb{N}$. This implies that $\mathcal{U}$ is not an eventual coloring within any $p \in \mathbb{N}$.

Now we have the following general problem for eventual coloring numbers.

Problem 9. *For each $n \geq 0$ and each $1 \leq k \leq n+1$, determine the minimal number $m_n(k)$ of natural numbers p satisfying the condition; if $f : X \to X$ is any fixed-point free homeomorphism of a separable metric space X such that* $\dim X = n$ *and* $\dim P(f) \leq 0$, *then $C(f, p) \leq n + 3 - k$.*

To prove the above results, we need the followings.

Lemma 10. *Let $\mathcal{C} = \{C_i|\ 1 \leq i \leq m\}$ be an open cover of a metric space X with $\dim X = n < \infty$ and let $\mathcal{B} = \{B_i|\ 1 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Suppose that O is an open set in X and Z is a zero-dimensional subset of O. Then there is an open shrinking $\mathcal{C}' = \{C_i'|\ 1 \leq i \leq m\}$ of $\mathcal{C}$ such that for each $i \leq m$,*

(0) $B_i \subset C_i'$,

(1) $C_i' = C_i$ *if* $bd(C_i) \cap O = \phi$,

(2) $C_i' \cap (X - O) = C_i \cap (X - O)$,

(3) $bd(C_i') \cap (X - O) \subset bd(C_i) \cap (X - O)$,

(4) $bd(C_i') \cap Z = \phi$, *and*

(5) $\{bd(C') \cap O|\ C' \in \mathcal{C}'\}$ *is in general position.*

Lemma 11. *Suppose that $f : X \to X$ is a fixed-point free homeomorphism of a metric space X such that* $\dim X = n < \infty$ *and* $\dim P(f) \leq 0$. *Let $\mathcal{C} = \{C_i |\ 1 \leq i \leq m\}$ be an open cover of X and let $\mathcal{B} = \{B_i |\ 1 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Then for any $k \in \mathbb{N}$, there is an open shrinking $\mathcal{C}' = \{C'_i |\ 1 \leq i \leq m\}$ of $\mathcal{C}$ such that*

(0) $B_i \subset C'_i$,

(1) $\{f^j(bd(C'))|\ C' \in \mathcal{C}', -k \leq j \leq k\}$ *is in general position,*

(2) $bd(C') \cap P(f) = \phi$ *for each* $C' \in \mathcal{C}'$.

Theorem 12. *Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. Then there exist a (metric) compactification αX of X and an extension $\alpha f : \alpha X \to \alpha X$ of f such that* $\dim \alpha X = \dim X$ *and* $\dim P_i(f) = \dim P_i(\alpha f)$ *for each $i \in \mathbb{N}$, where*

$$P_i(f) = \{x \in X \mid f^j(X) = x \text{ for some } j \ (1 \le j \le i)\}.$$

Corollary 13. *Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. If $P(f) = \phi$, then there exist a (metric) compactification αX of X and an extension $\alpha f : \alpha X \to \alpha X$ of f such that* $\dim \alpha X = \dim X$ *and $P(\alpha f) = \phi$.*

Remark 14. *Let X be a finite-dimensional metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. Then the extension $\beta f : \beta X \to \beta X$ of f over the Cech-Stone compactification βX satisfies the condition $Cl_{\beta X}(P_i(f)) = P_i(\beta f)$ for each $i \in \mathbb{N}$.*

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