Selections, games and metrisability of manifolds

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TE PŪTEA RANGAHAU
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M^m a manifold (=T_2, connected, locally euclidean). **Spot the odd one out!**

- M embeds properly in some euclidean space;
- M is completely metrisable;
- M is metrisable;
- M is second countable;
- M is separable;
- M is Lindelöf;
- M is paracompact;
- M is nearly linearly ω_1-metaLindelöf;
- M has a k-network which is point-countable on a dense subset;
- the tangent microbundle on M is equivalent to a fibre bundle;
- M has a regular G_δ-diagonal;
- there is a surjective immersion $f : \mathbb{R}^m \to M$;
- M satisfies the selection principle $\mathsf{S}_1(\mathcal{K}, \Gamma)$;
- Player K has a winning strategy in Gruenhage's game $G^o_{K,L}(M)$;
- $C_k(M; \mathbb{R})$ is Volterra;
- $\mathcal{H}(M)$ is metrisable.

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$$\mathbb{S}_1(\mathcal{K},\mathcal{K})$$

$\mathcal{K}$ = all k-covers of X: open covers $\mathcal{U}$ with $X \notin \mathcal{U}$ and every compactum of X is in some member of $\mathcal{U}$

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$$\mathbb{S}_1(\mathcal{K},\mathcal{K}) \underset{(1^{\text{st}}\text{ countable})}{\iff} X \text{ is hemicompact}$$

(Caserta, Di Maio, Kocinac, Meccariello 2006)

$$\underset{(\text{manifold})}{\iff} X \text{ is metrisable (G, Mynard 2005)}$$

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$\boxed{\mathbb{S}_1(\mathcal{K},\mathcal{K})}$

For a manifold

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For a manifold

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(McCoy 1980)
$C_k(X)$ is a k-space $\Longleftrightarrow$
$\Longleftrightarrow$ X is metrisable
(manifold) (G, Mynard 2005)

$$
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For a manifold

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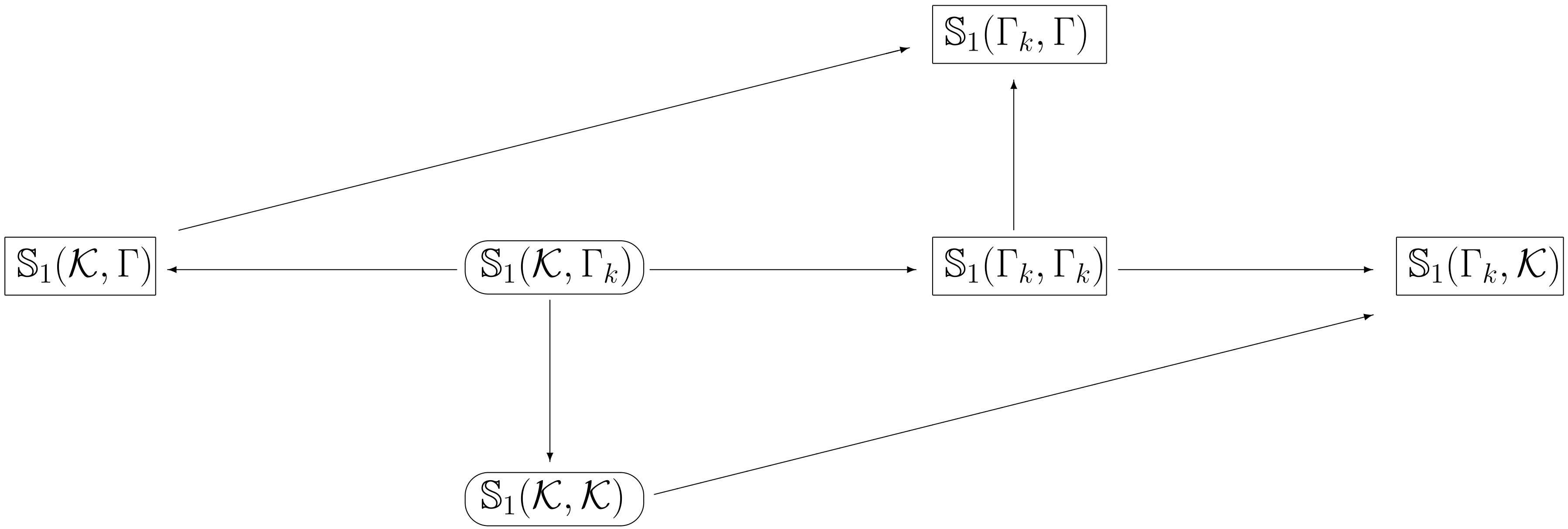

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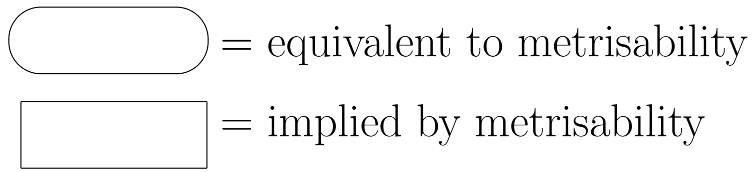

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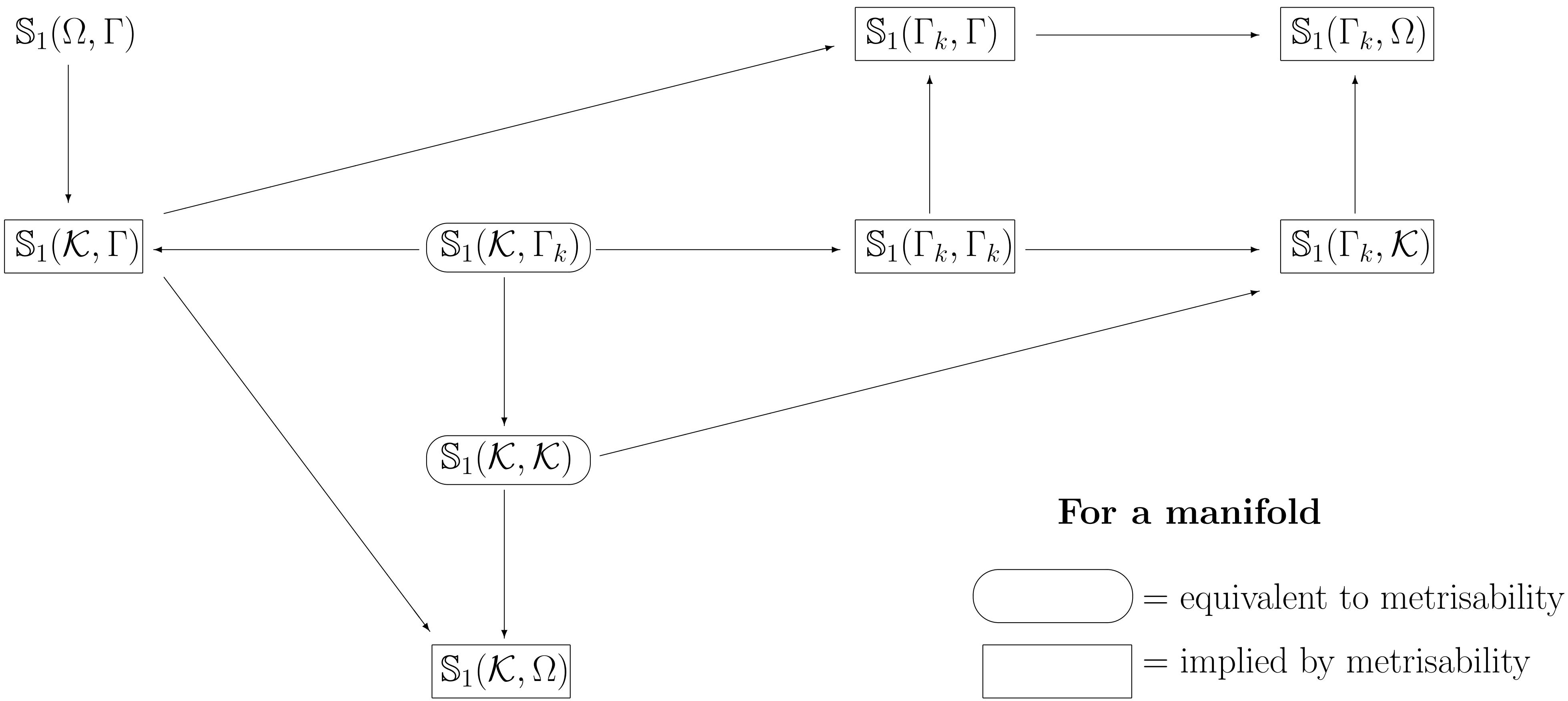

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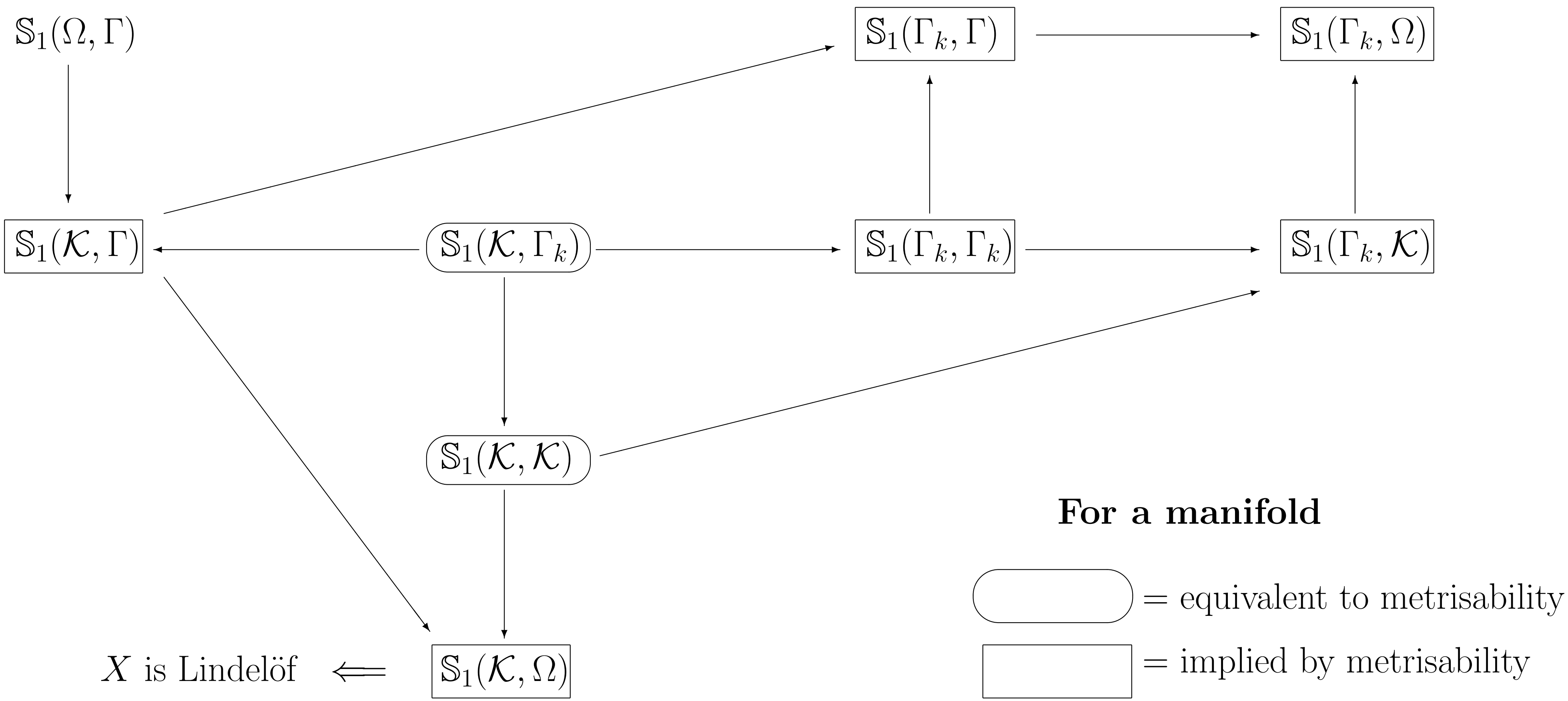

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$$X \text{ satisfies } \mathbb{S}_1(\mathcal{K}, \Omega) \Longrightarrow X \text{ Lindelöf.}$$

$\mathcal{U}$ open cover of X.

$$\widehat{\mathcal{U}} = \{U_1 \cup \ldots U_n \ / \ U_i \in \mathcal{U} \text{ and } n \in \mathbb{N}\}: \quad \widehat{\mathcal{U}} \in \mathcal{K}$$

Apply $\mathbb{S}_1(\mathcal{K}, \Omega)$ to $\langle \widehat{\mathcal{U}} \rangle$:

$$\rightsquigarrow \langle V_n \rangle, V_n \in \widehat{\mathcal{U}}: \quad \{V_n \ / \ n \in \mathbb{N}\} \in \Omega.$$

Each V_n a finite union of members of $\mathcal{U}$: collect together

$$\rightsquigarrow \text{countable subcover of } \mathcal{U}.$$

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X is metrisable $\overset{\text{(manifold)}}{\Longleftrightarrow}$

X is Lindelöf $\Longleftarrow$ $\boxed{\mathbb{S}_1(\mathcal{K},\Omega)}$

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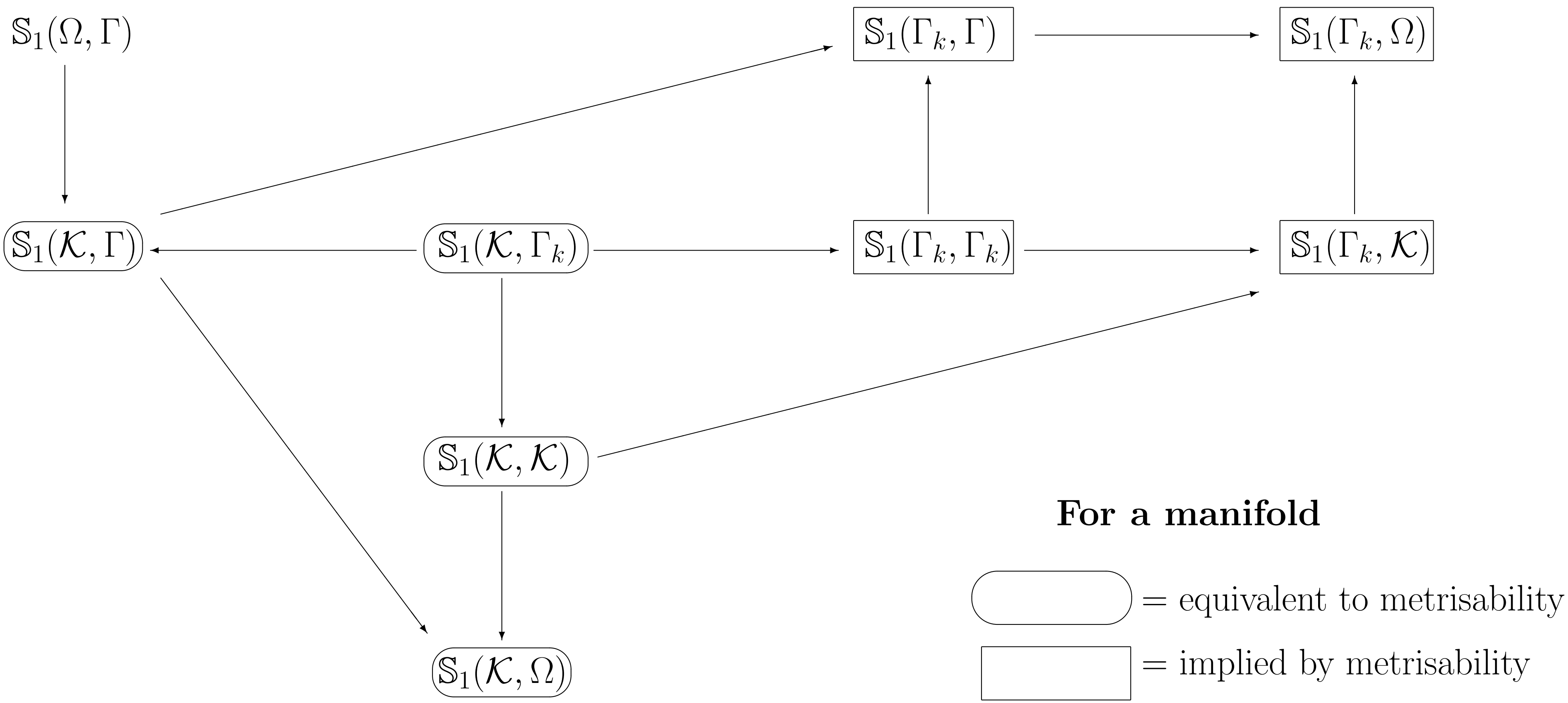

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K wins iff $\langle L_n \rangle_{n\in\omega}$ has a discrete open expansion:

$\exists$ sequence $\langle U_n \rangle_{n\in\omega}$ open sets such that

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(G, Mynard 2006) Manifold M metrisable $\Longrightarrow C_k(M)$ Polish.

(Kechris 1995) $C_k(M)$ Polish $\Longrightarrow C_k(M)$ weakly α-favourable.

(Gruenhage 2006) $C_k(M)$ weakly α-favourable $\Longrightarrow$ K has a winning strategy in $G^o_{K,L}(M)$.

(Gruenhage 2006) K has a winning strategy in $G^o_{K,L}(X) \Longrightarrow X$ is paracompact.

(Смирнов 1951) M is paracompact $\Longrightarrow M$ is metrisable.

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For ordinal $\beta < \alpha$

- Player One chooses $\mathcal{A}_\beta \in \mathbb{A}$;
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Player Two wins play $\mathcal{A}_0, \mathcal{T}_0, \ldots, \mathcal{A}_\beta, \mathcal{T}_\beta, \ldots$ iff $\cup_{\beta<\alpha}\mathcal{T}_\beta \in \mathbb{B}$.

(Babinkostova 2005) showed that if metrisable X has covering dimension at most n then Player Two has a winning strategy in $\mathsf{G}_c^{n+1}(\mathbb{O}, \mathbb{O})$. An n-manifold has covering dimension n.

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$$
\begin{aligned}
& \text{Player Two has a winning strategy in } \mathsf{G}_c^{n+1}(\mathbb{O}, \mathbb{O}) \\
\Longrightarrow\ & \text{Player Two has a winning strategy in } \mathsf{G}_c^{\omega}(\mathbb{O}, \mathbb{O}) \\
\Longrightarrow\ & X \text{ is metaLindelöf} \\
\Longrightarrow\ & X \text{ is metrisable provided also that } X \text{ is connected and locally second countable.}
\end{aligned}
$$