UDK 515.122

The Lebesgue and inductive dimensions defined by means of m-coverings

Vitaly V. Fedorchuk

Moscow State University

1. Introduction

In [2] there were introduced classes of $\mathcal{G}$-C-*spaces* and m-$\mathcal{G}$-C-space, where $\mathcal{G}$ is a class of simplicial complexes and $m \geq 2$ is an integer. Partial cases of these classes were considered in [3], where (m,n)-C-spaces were defined ($m \geq n \geq 1$). Let (m,n)-C be the class of all (m,n)-C-spaces. Then all classes (m,n)-C are intermediate between the class wid $=(2,1)$-$C = (n+1,n)$-C of all *weakly infinite-dimensional spaces* in the sense of Smirnov and the class C of all C-spaces in the sense of Haver [4], Addis and Gresham [1]. For example,

$$\text{wid} = (2,1)\text{-}C \supset (3,1)\text{-}C \supset \ldots \supset (m,1)\text{-}C \supset \ldots \supset C.$$

Here we define and investigate new dimension functions: (m,n)-dim and (m,n)-Ind.

2. Definitions

All spaces are assumed to be normal T_1. All mappings are continuous. The symbol $|A|$ stands for the cardinality of a set A. If A is a subset of a space X, then $\mathrm{Cl}(A) = \mathrm{Cl}_X(A)$ denotes the closure of A in X.

By a cover we mean an open cover of a space. By $\mathrm{cov}(X)$ we denote the set of all covers of X. The set of all finite covers of X is denoted by $\mathrm{cov}_\infty(X)$ and $\mathrm{cov}_m(X)$ stands for the set of all covers of X consisting of $\leq m$ members.

Let u and v be families of subsets of a set X. They say that v *refines* u (v is a *refinement* of u) if each $V \in v$ is contained in some $U \in u$. A family v *combinatorially refines* u (v is a *combinatorial refinement* of u) if there exists an injection $i : v \to u$ such that $V \subset i(V)$ for each $V \in v$. If v refines u we write $u \prec v$.

For a finite simplicial complex K by $v(K)$ we denote the set of all its vertices. By $\mathrm{Fin}S$ we denote the set of all non-empty finite subsets of S. Let u be a family of arbitrary sets and let $u_0 = \{U \in u : U \neq \emptyset\}$. The *nerve* $N(u)$ of the family u is a simplicial complex such that $v\Big(N(u)\Big) = \{a_U : U \in u_0\}$ and a set $\triangle \in \mathrm{Fin}v\Big(N(u)\Big)$ is a simplex of $N(u)$ if and only if $\bigcap\{U : a_U \in \triangle\} \neq \emptyset$.

By the *order* of a family u of sets we mean the largest n such that u contains n sets with non-empty intersection. If no such integer exist, we say that u has order ∞. The order of u is denoted by $\mathrm{ord}u$. Clearly,

$$\mathrm{ord}u \leq n \iff \dim N(u) \leq n-1;$$

$$\mathrm{ord}u \leq 1 \iff u \quad \text{is a disjoint family.}$$

By $\mathbb{N}$ we denote the set of all positive integers.

Let u be a family of subsets of a set X and let $M \subset X$. Then

$$u|M = \{U \cap M : U \in u\}.$$

2.1. Definition. Let $u = (U_1, \ldots, U_m) \in \mathrm{cov}_m(X)$ and let $\Phi = (F_1, \ldots, F_m)$ be a sequence of closed subsets of X such that

$$F_j \subset U_j, \ j = 1, \ldots, m;$$

$$\mathrm{ord}\Phi \leq 1.$$

Then (u, Φ) is said to be an *m-pair in X*. The set of all m-pairs in X is denoted by $m(X)$.

2.2. Definition. Let $m, n \in \mathbb{N}$, $n \le m$, (u, Φ) be an m-pair in X and let $v = (V_1, \ldots, V_m)$ be a family of open subsets of X such that

$$F_j \subset V_j \subset U_j, \; j = 1, \ldots, m :$$

$$\mathrm{ord} v \le n.$$

Then (u, v, Φ) is called an (m, n)-*triple in* X.

2.3. Definition. Let $(u, \Phi) \in m(X)$. A closed set $P \subset X$ is said to be an n-*partition* of (u, Φ) (notation: $P \in \mathrm{Part}(u, \Phi, n)$) if there exists an (m, n)-triple (u, v, Φ) in X such that $P = X \setminus \bigcup v$.

2.4. Definition. Let $(u_i, \Phi_i) \in m(X)$, $i = 1, \ldots, r$. The sequence $\big((u_1, \Phi_1), \ldots, (u_r, \Phi_r)\big)$ is called n-*inessential in* X if there exist n-partitions $P_i \in \mathrm{Part}(u_i, \Phi_i, n)$ such that $P_1 \cap \ldots \cap P_r = \emptyset$.

2.5. Definiton. Let $m, n \in \mathbb{N}, n \leq m$. To every space X one assigns the *dimension* (m, n)-dim$X$, which is an integer ≥ -1 or ∞. The dimension function (m, n)-dim is defined in the following way:

(1) (m, n)-dim$X = -1$ if and only if $X = \emptyset$;

(2) (m, n)-dim$X \leq k$, where $k = 0, 1, \ldots,$ if every sequence $\Big((u_1, \Phi_1), \ldots, (u_{k+1}, \Phi_{k+1})\Big)$, $(u_i, \Phi_i) \in m(X)$, is n-inessential in X;

(3) (m, n)-dim$X = \infty$, if (m, n)-dim$X > k$ for each $k \in \mathbb{N}$.

2.6. Definition. Let $m, n \in \mathbb{N}$. To every space X one assings the *inductive dimension* (m, n)-Ind$X$, which is an integer ≥ -1 or ∞. The dimension function (m, n)-Ind is defined in the following way:

(1) (m, n)-Ind$X = -1$ if and only if $X = \emptyset$;

(2) (m, n)-Ind$X \leq k \geq 0$ if for every pair $(u, \Phi) \in m(X)$ there exists an n-partition $P \in \mathrm{Part}(u, \Phi, n)$ such that (m, n)-Ind$P \leq k - 1$;

(3) (m, n)-Ind$X = \infty$ if (m, n)-Ind$X > k$ for $k = -1, 0, 1, \ldots$.

3. Dimension (m, n)-dim

From theorem on partitions we get

3.1. Theorem. *For every space X we have*

$$(2,1)\text{-dim}X = \text{dim}X.$$

The next statement is rather clear.

3.2. Proposition. *Let M be a closed subset of X. Then*

$$(m,n)\text{-dim}M \leq (m,n)\text{-dim}X.$$

3.3. Proposition. *If a space X can be represented as the union of a discrete family X_α, $\alpha \in A$, of closed subspaces such that (m,n)-dim$X_\alpha \leq k$ for $\alpha \in A$, then (m,n)-dim$X \leq k$.*

3.4. Definition. For a subspace M of a space X, the *relative* (m,n)-*dimension* of M is defined by the formula

$$r\text{-}(m,n)\text{-}d_X M = \sup\{(m,n)\text{-}\dim F : F \subset M \ \ \textit{and} \ \ F \ \ \textit{is closed in} \ \ X\}.$$

Proposition 3.2 implies

3.5. Proposition. *For every normal subspace M of a space X we have*

$$r\text{-}(m,n)\text{-}d_X M \leq (m,n)\text{-}\dim M.$$

3.6. Proposition. *If a hereditarily normal space X can be represented as the union of two subspaces Y and Z such that*

$$(m,n)\text{-}\dim Y \leq k, \quad r\text{-}(m,n)\text{-}d_X Z \leq l$$

then

$$(m,n)\text{-}\dim X \leq k + l + 1.$$

Proposition 3.6 implies

3.7. The addition theorem for (m,n)-dim. *If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then*

$$(m,n)\text{-dim}X \leq (m,n)\text{-dim}X_1 + (m,n)\text{-dim}X_2 + 1.$$

Theorem 3.7 yields

3.8. Corollary. *If a hereditarily normal space X can be represented as the union of $k+1$ subspaces $X_0, X_1, \ldots, X_k$ such that (m,n)-dim$X_i \leq 0$ for $i = 0, 1, \ldots, k$, then (m,n)-dim$X \leq k$.*

3.9. Theorem. *Let $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ be an inverse system of compact spaces X_α with (m,n)-dim$X_\alpha \leq k$, and let $X = \lim S$. Then (m,n)-dim$X \leq k$.*

3.10. Theorem. *If $n < m$, then for every space X we have*

$$(m,n)\text{-dim}X \leq 0 \iff \text{dim}X \leq n-1.$$

4. Valuations for dimension (m,n)-dim$X$

4.1.Definition[3]. A space X is called an $(m,n)-C$-*space* (notation: $X \in (m,n)-C$) if for every sequence $u_i \in cov_m(X), i \in \mathbb{N}$, there exist a sequence v_i of families of open subsets of X such that v_i refines u_i, ord$v_i \leq n$ and $\cup\{v_i : i \in \mathbb{N}\} \in$ cov(X).

The following statement is an immediate consequence of definitions.

4.2. Proposition. *If* (m,n)-dim$X < \infty$, *then* $X \in (m,n) - C$.

Theorem 3.6 and Proposition 3.7 from [3] yield

4.3. Theorem. *Every compact space* $X \in (m,n)$-C *is weakly infinite-dimensional.*

4.4. Theorem. *For every spce X,*

$$n\Big((m,n)\text{-dim}\Big) \leq \text{dim}X.$$

Theorem 4.4. implies

4.5. Theorem. *For an arbitrary space X we have*

$$(m,n)\text{-dim}X \leq \text{dim}X.$$

4.6. Theorem. *If $\Big((m,n)\text{-dim}X + 1\Big)n < m$, then*

$$\text{dim}X \leq n\Big((m,n)\text{-dim}\Big) + n - 1.$$

Theorems 4.4 and 4.6 yield

4.7. Corollary. *If* $\big((m,n)\text{-dim}X+1\big)n<m$, *then*

$$\frac{\text{dim}X}{n}-1+\frac{1}{n}\leq (m,n)\text{-dim}X\leq \frac{\text{dim}X}{n}.$$

4.8. Question. Is the assertion of Theorem 4.6 true unconditionally?

4.9. Theorem. *For every* $r\geq 0$ *there exists a metrizable separable space* X_r *such that*

$$(m,n)\text{-dim}X_r=r.$$

4.10. Question. Is it true that for every $r\geq 0$ there exists a metrizable compact space Z_r such that

$$(m,n)\text{-dim}Z_r=r?$$

5. Basic properties of dimensions (m,n)-Ind

From definition of the large inductive dimension Ind we get

5.1. Theorem. *For every space X we have*

$$(2,1)\text{-Ind}X = \text{Ind}X.$$

5.2. Proposition. *If Y is closed in X, then*

$$(m,n)\text{-Ind}Y \leq (m,n)\text{-Ind}X.$$

5.3. Proposition. *Let X be the discrete union of subspaces X_α, $\alpha \in A$. Then (m,n)-Ind$X \leq k$ if and only if (m,n)-Ind$X_\alpha \leq k$ for every $\alpha \in A$.*

5.4. Theorem. *For every space X we have*

$$(m,n)\text{-Ind}X = (m,n)\text{-Ind}\beta X.$$

5.5. Theorem. *Let X be a hereditarily normal space and let $X = X_1 \cup X_2$. Then*

$$(m,n)\text{-Ind}X \leq (m,n)\text{-Ind}X_1 + (m,n)\text{-Ind}X_2 + 1.$$

5.6. Corollary. *If a hereditarily normal space X can be represented as the inion of $k+1$ subspaces $X_0, X_1, \ldots, X_k$ so that (m,n)-Ind$X_i \leq 0$ for $i = 0, 1, \ldots, k$, then (m,n)-Ind$X \leq k$.*

5.7. Proposition. *For every space X we have*

$$(m,n)\text{-dim}X \leq 0 \iff (m,n)\text{-Ind}X \leq 0.$$

5.8. Theorem. *For every space X we have*

$$(m,n)\text{-dim}X \leq (m,n)\text{-Ind}X.$$

5.9. Theorem. *Let* Ind$X \leq r$. *If* Ind *satisfies a finite* sum *theorem for dimensions $\leq r-1$, then (m,n)-Ind$X \leq$ Ind$X$.*

As a corollary we get

5.10. Theorem. *For an arbitrary space* X *with* (m,n)-Ind$X \leq 1$ *we have*

$$(m,n)\text{-Ind}X \leq \text{Ind}X.$$

5.11. Theorem. *If* X *is a perfectly normal space, then* (m,n)-Ind$X \leq$ Ind$X$.

5.12. Theorem. *If* X *is a perfectly normal space, then* $(m,1)$-Ind$X =$ Ind$X$.

5.13. Question. Does the equality $(m,1)$-Ind $=$ Ind$X$ hold for an arbitrary space X?

References

[1] D.F.Addis and J.H.Gresham, A class of infinite-dimensional spaces. I. Dimension theory and Alexandroff's problem. Fund. Math., **101:3** (1978), 195–205.

[2] V.V.Fedorchuk, C-spaces and simplicial complexes, Sib. Matem. J., **50:4** (2009), 933–941 (in Russian).

[3] V.V.Fedorchuk, C-spaces and matrix of infinite-dimensionality, Topol. Appl., **157:17** (2010), 2622-2634.

[4] W.E.Haver, A covering property for metric spaces. Lecture Notes in Math., **375** (1974), 108–113.

Thank You!