Indestructibility of compact spaces

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Indestructible Lindelöf spaces

Definition (Tall 1995)

A Lindelöf space is *indestructible* if it remains Lindelöf after forcing with any countably closed partial order.

Theorem (Scheepers-Tall 2010)

A Lindelöf space X is indestructible if and only if player One does not have a winning strategy in the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ — in short, One $\not\uparrow \mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$.

$\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ denote, respectively, the length-ω_1 versions of the Rothberger property $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ and the Rothberger game $\mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X)$.

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$\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ *and* $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$

As usual, if One $\not\uparrow \mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ then $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ holds.

Theorem (Pawlikowski 1994)

$\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ is equivalent to One $\not\uparrow \mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X)$.

Question (Scheepers-Tall 2010)

Is $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ equivalent to One $\not\uparrow \mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$?

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A counterexample

Question (Scheepers-Tall 2010)

Is $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X,\mathcal{O}_X)$ equivalent to One $\not\uparrow$ $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X,\mathcal{O}_X)$?

Answer: No, if CH is assumed.

Example

Consider the lexicographical ordering on the set $Y = {}^{\omega_1}2$, and regard Y as a linearly ordered topological space. Then

- Y is compact;
- One $\uparrow$ $\mathsf{G}_1^{\omega_1}(\mathcal{O}_Y,\mathcal{O}_Y)$ — *i.e.*, Y is destructible;
- CH implies $\mathsf{S}_1^{\omega_1}(\mathcal{O}_Y,\mathcal{O}_Y)$.

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Definition

Let κ be an infinite cardinal. A *κ-Čech-Pospíšil tree* in a topological space X is an indexed family $\langle F_s : s \in {}^{\leq\kappa}2\rangle$ satisfying:

(*i*) each F_s is a nonempty closed subset of X;
(*ii*) $F_s \supseteq F_t$ whenever $s \subseteq t$;
(*iii*) $F_{s^\frown(0)} \cap F_{s^\frown(1)} = \emptyset$.

Proposition

If there is an ω_1-Čech-Pospíšil tree in X, then One $\uparrow \mathsf{G}_1^{\omega_1}(\mathcal{O}_X,\mathcal{O}_X)$.

Since no point of Y is a G_δ, we have:

Corollary

Y is destructible.

$S_1^{\omega_1}(\mathcal{O},\mathcal{O})$ *and* $G_1^{\omega_1}(\mathcal{O},\mathcal{O})$

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CH implies $\mathsf{S}_1^{\omega_1}(\mathcal{O}_Y,\mathcal{O}_Y)$.

Proof.

It suffices to show that there is $C \subseteq Y$ with $|C| = \aleph_1$ satisfying:

For every open $U \subseteq Y$ such that $C \subseteq U$, we have $|Y \setminus U| \leq \aleph_1$.

Let $C = \{f \in Y : f \text{ is eventually constant}\}$. Note that $|C| = \aleph_1$ by CH.

Claim. *If $U \subseteq Y$ is open and $C \subseteq U$, then $Y \setminus U$ is finite.*

Suppose not. Since $F = Y \setminus U$ is compact, any countable infinite $A \subseteq F$ has an accumulation point $p \in F$. This implies that p must be an element of C, which is a contradiction since $C \cap F = \emptyset$. $\square$

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Indestructibility of dyadic spaces

Lemma

The following conditions are equivalent for an infinite cardinal κ:

(*a*) 2^κ is indestructible;

(*b*) $\mathsf{S}_1^{\omega_1}(\mathcal{O}_{2^\kappa}, \mathcal{O}_{2^\kappa})$;

(*c*) $\kappa = \omega$.

Corollary

The following are equivalent for a dyadic space X:

(*a*) X is indestructible;

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(*d*) $w(X) = \aleph_0$.

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ω_1-dyadic systems

Theorem (Šapirovskiĭ 1980)

The following conditions are equivalent for a compact T_2 space X and an uncountable cardinal κ:

(*a*) there is a closed nonempty $F \subseteq X$ such that $\pi\chi(x, F) \geq \kappa$ for all $x \in F$;

(*b*) there is a κ-dyadic system in X, *i.e.* an indexed family $\langle F^i_\alpha : \alpha \in \kappa, i \in 2 \rangle$ of closed subsets of X such that $F^0_\alpha \cap F^1_\alpha = \emptyset$ for all $\alpha \in \kappa$ and $\bigcap\{F^{p(\xi)}_\xi : \xi \in \mathrm{dom}(p)\} \neq \emptyset$ for all $p \in Fn(\kappa, 2)$.

Note that the existence of an ω_1-dyadic system in a compact space X is equivalent to the failure of $\mathrm{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ witnessed by a sequence of 2-element open covers of X.

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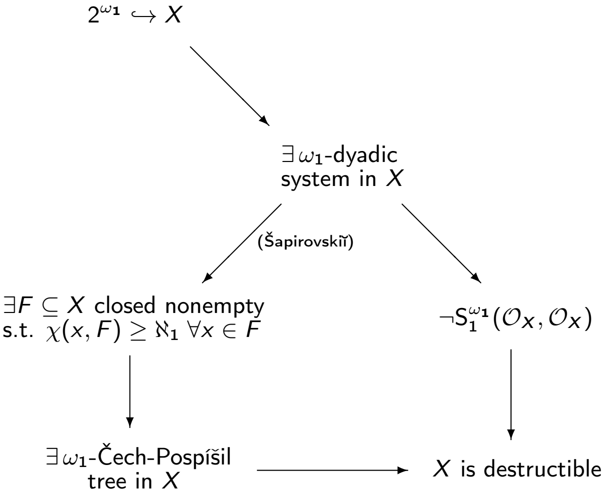
The big picture

for compact Hausdorff spaces



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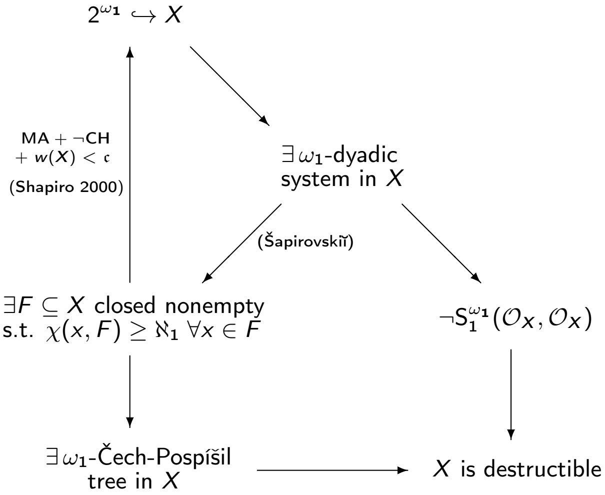

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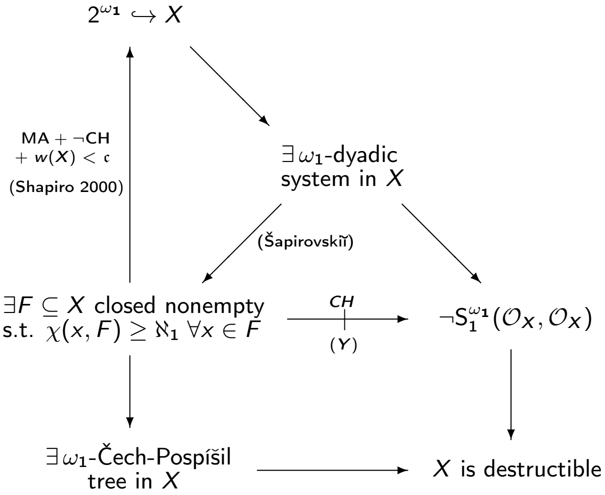

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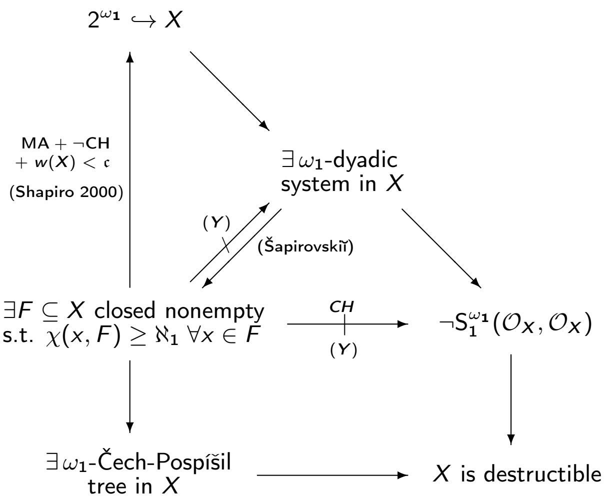

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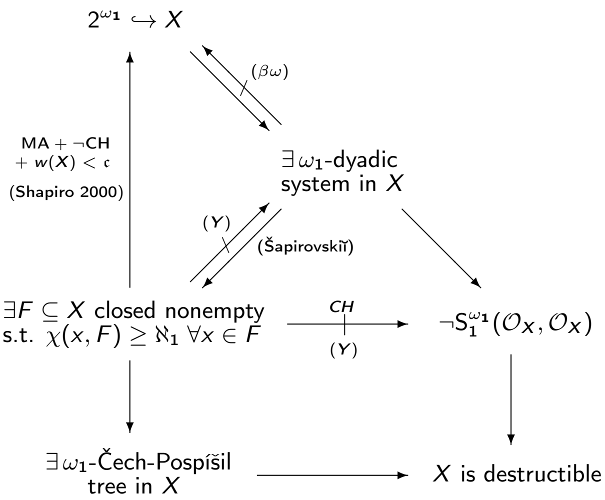

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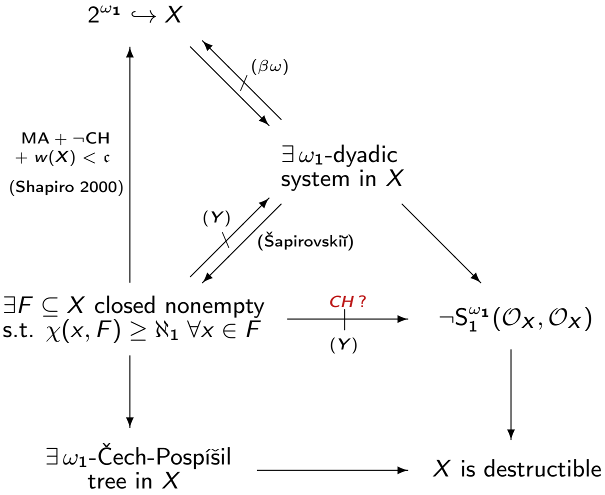

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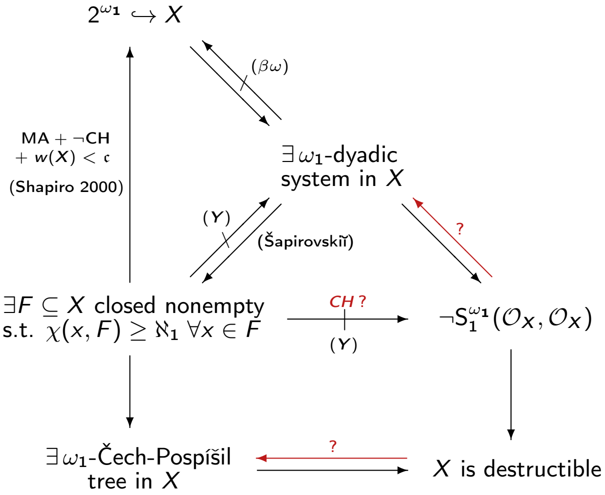

Questions

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Is there a ZFC example of a destructible space satisfying $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$?

Question 2

For X compact, is $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ equivalent to the nonexistence of an ω_1-dyadic system in X?

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Is destructibility of a space X equivalent to the existence of an ω_1-Čech-Pospíšil tree in X?

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Grazie mille!

References

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