CERTAIN TYPES OF OPEN COVERS AND SELECTION PRINCIPLES USING IDEALS

PRATULANANDA DAS

1. Background

We start by recalling the definition of natural or asymptotic density as follows: If N denotes the set of natural numbers and $K \subset N$ then $K(n)$ denotes the set $\{k \in K : k \leq n\}$ and $|K(n)|$ stands for the cardinality of the set $K(n)$. The natural or asymptotic density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$$

provided the limit exists.

It can be mentioned in this context that using the idea of asymptotic density, the idea of convergence of a real sequence had been extended to statistical convergence by Fast [6] (see also Schoenberg [22]) as follows: A sequence $\{x_n\}_{n\in N}$ of points in a metric space (X, ρ) is said to be statistically convergent to ℓ if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in N : d(x_k, \ell) \geq \epsilon\}$ has natural density zero. A lot of investigations have been done on this convergence and its topological consequences after the initial works by Fridy [7] and Connor [3]. The most prominent being the work of Di Maio and Kočinac [5].

On the other hand, in [16] an interesting generalization of the notion of statistical convergence was proposed. Namely it is easy to check that the family $I_d = \{A \subset N : d(A) = 0\}$ forms a non-trivial admissible ideal of N (recall [17], [18] that if X is a nonempty set then a family I of subsets of X is said to be an *ideal* in X if (i) $\phi \in I$, (ii) $A, B \in I$ implies $A \cup B \in I$ and (iii) $A \in I$, $B \subset A$ implies $B \in I$. A nonempty family F of subsets of X is called a *filter* on X if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$, (iii) $A \in F$ and $A \subset B$ implies $B \in F$. I is called *non-trivial* if $I \neq \{\phi\}$ and $X \notin I$. If I is a proper nontrivial ideal then the family of sets $F(I) = \{M \subset X :$ there exists

$A \in I : M = X \setminus A\}$ is a filter in X. It is called the *dual filter* of the ideal I. A proper ideal I is called *admissible* if $\{x\} \in I$ for each $x \in X$. Such ideals are also called *free* ideals.) Thus one may consider an arbitrary ideal I of N and define I-convergence of a sequence by replacing the sets of density zero by the members of the ideal.

2. Main Results

Throughout (X, τ) stands for a Hausdorff topological space. $\mathbb{F}(X)$ will stand for the class of nonempty finite sets.

We now write down two classical selection principles formulated in general form in [19] (see also [10], [4]). For two nonempty classes of sets $\mathbf{A}$ and $\mathbf{B}$ of an infinite set X we define:

$S_1(\mathbf{A}, \mathbf{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathbf{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each n and $\{b_n : n \in \mathbb{N}\} \in \mathbf{B}$.

$S_{fin}(\mathbf{A}, \mathbf{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathbf{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ of finite (possibly empty) sets such that $B_n \subset A_n$ for each n and $\bigcup_n B_n \in \mathbf{B}$.

There are infinitely long games corresponding to these selection principles.

$G_1(\mathbf{A}, \mathbf{B})$ denotes the game for two players, ONE and TWO, who play a round for each positive integer n. In the nth round ONE chooses a set A_n from $\mathbf{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins the play $(A_1, b_1, \dots, A_n, b_n, \dots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathbf{B}$; otherwise ONE wins.

$G_{fin}(\mathbf{A}, \mathbf{B})$ denotes the game where in the nth round ONE chooses a set A_n from $\mathbf{A}$ and TWO responds by choosing a finite (possibly empty) set $B_n \subset A_n$. TWO wins the play $(A_1, B_1, \dots, A_n, B_n, \dots)$ if $\bigcup_n B_n \in \mathbf{B}$ otherwise ONE wins.

In [14] following selection principles were introduced and studied (see also [24], [5]).

The symbol $\alpha_i(\mathbf{A}, \mathbf{B})$, $i = 1, 2, 3, 4$denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of infinite elements from $\mathbf{A}$ there is an element $B \in \mathbf{B}$ such that:

$\alpha_1(\mathbf{A}, \mathbf{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;

$\alpha_2(\mathbf{A}, \mathbf{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_3(\mathbf{A}, \mathbf{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_4(\mathbf{A}, \mathbf{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

We now recall some classes of open covers which we will use throughout the paper. If Δ is a collection of subsets of the topological space X then an open cover $\mathcal{U}$ of X is called a Δ *cover* if X does not belong to $\mathcal{U}$ and every member of Δ is contained in a member of $\mathcal{U}$. $\mathcal{O}_\Delta$ will denote the family of all Δ covers. When Δ is $\mathbb{F}(X)$ the Δ covers are called *ω-covers*, and we use the symbols Ω to denote the set of ω-covers. A countable Δ-cover $\mathcal{U}$ is called *groupable* (or Δ-*groupble*) if it can be represented as a countable union of finite families $\mathcal{U}_n, n \in \mathbb{N}$, where $\mathcal{U}_m \cap \mathcal{U}_n = \phi$ whenever $m \neq n$ such that for each $D \in \Delta$, for all but finitely many n there is $U \in \mathcal{U}_n$ such that $D \subset U$. The symbol $\mathcal{O}_\Delta^{gp}$ is used to denote the set of Δ-groupable covers. For $\Delta = \mathbb{F}(X)$, groupable Δ-covers are called *goupable ω-covers*, and for Δ the family of singletons we use the term groupability only (see [15]). The sets of groupable and ω-groupable covers will be denoted by $\mathcal{O}^{gp}$ and Ω^{gp} respectively. A Δ-cover $\mathcal{U}$ is called a γ_Δ-*cover* if for each $D \in \Delta$ the set $\{U \in \mathcal{U} : D \not\subseteq U\}$ is finite. The symbol Γ_Δ denotes the set of all γ_Δ-covers. $\gamma_{\mathbb{F}(X)}$-covers are called γ-covers and the set of such covers is denoted by Γ.

Throughout I will stand for a proper ideal of $\mathbb{N}$.

We now introduce the following definitions.

Let Δ be a family of subsets of a space X. A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be:

1. An $I-\gamma$-*cover* ($I-\gamma_\Delta$-cover) if for each $x \in X$ (for each $D \in \Delta$) the set $\{n \in \mathbb{N} : x \notin U_n\}$ ($\{n \in \mathbb{N} : D \nsubseteq U_n\}$) belongs to I.

2. An I-*groupable cover* ($I - \Delta$-*groupable cover*) if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{V}_n, n \in \mathbb{N}$, such that for each $x \in X$ (for each $D \subset \Delta$) the set $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\}$ ($\{n \in \mathbb{N} : D \nsubseteq V$ for every $V \in \mathcal{V}_n\}$) belongs to I.

We denote the set of all $I - \gamma$-covers ($I - \gamma_\Delta$-covers, I-groupable covers, $I-\Delta$-groupable covers) by $I-\Gamma$ ($I-\Gamma_\Delta$, $I - \mathcal{O}^{gp}$, $I - \mathcal{O}^{gp}_\Delta$). For the ideal of all finite subsets of $\mathbb{N}$, $I = I_{fin}$ we get the standard notions of γ-covers (γ_Δ-covers, groupable covers, Δ-groupable covers etc) and for $I = I_d$, the ideal of zero density sets, we get the statistical variants of these notions [5].

Lemma 2.1. *An open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of a topological space* X *is an* $I-\gamma$*-cover of* X *if and only if for each finite set* $F \subset X$ *the set* $\{n \in \mathbb{N} : F \nsubseteq U_n\} \in I$*.*

Evidently every γ-cover of X is also an $I-\gamma$-cover of X when I is admissible but the converse is not true.

Example 2.1. *Let* $X = \mathbb{R}\backslash\{0\}$*, the set of all nonzero real numbers with the usual topology. Let* I *be an admissible ideal of* $\mathbb{N}$ *and* $I \neq I_{fin}$ *the ideal of all finite subsets of* $\mathbb{N}$*. Then* I *must contain an infinite set* $A = \{m_1 < m_2 < m_3, \dots\}$ *(say). Without any loss of generality take* $m_1 > 1$*. Put* $U_1 = (-\infty, 0)$ *and* $U_{m_i} = (0, m_i)$ *for all* $i \geq 1$ *and* $U_i = X$ *for all* i*,* $i \neq m_j$ *for any* j*. Consider the open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* X*. Then* $\mathcal{U}$ *is an* $I-\gamma$*-cover of* X*. If* $x \in (-\infty, 0)$ *then* $x \notin U_{m_i}$ *for all* $i \in \mathbb{N}$ *but the set* $\{m_i : i \in \mathbb{N}\} \in I$*. If* $x \in [0, \infty)$ *then we can always choose a* $m_j > x$ *and so* x *does not belong to* U_1 *and at most finite number of* $U_{M_i}s$ *and the result follows from the fact that* I *is admissible.*

On the other hand $\mathcal{U}$ is clearly not a γ-cover of X since for any $x \in (-\infty, 0)$, x does not belong to infinite number of members of $\mathcal{U}$.

It is well known that any infinite subset of a γ-cover is also a γ-cover but definitely this is not true for $I-\gamma$-covers. However we can prove the following.

We call a subset $\mathcal{V}$ of a cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X I-dense in $\mathcal{U}$ if the set $M = \{m_1 < m_2 < m_3 < \dots\}$ of indices of elements from $\mathcal{V}$ belongs to $F(I)$ and further if $f : \mathbb{N} \to M$ be the bijection given by $f(i) = m_i$ then $f(A) \in I$ if and only if $A \in I$.

Lemma 2.2. *An I-dense subset of an $I-\gamma$-cover of X is also an $I-\gamma$-cover of X.*

Lemma 2.3. *Let $(\mathcal{U}_n : n \in \mathbb{N})$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a sequence of (countable) $I-\gamma$-covers of X. Then $(\mathcal{V}_n : n \in \mathbb{N})$, defined by*

$$\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \dots \cap U_{n,m} : m \in \mathbb{N}\} \setminus \{\phi\}$$

is also a sequence of $I-\gamma$-covers of X.

Recall that a space X is said to have the *Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of

X there are finite $\mathcal{V}_n \subset \mathcal{U}_n, n \in \mathbb{N}$ such that each $x \in X$ belongs to all but finitely many sets $\bigcup \mathcal{V}_n$ [9].

We can generalize this concept (and subsequently the notion of s-Hurewicz property [5]) with the help of ideals and say that a space X is said to have the *I-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there are finite $\mathcal{V}_n \subset \mathcal{U}_n, n \in \mathbb{N}$ such that each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in I$.

If F is a filter then following the line of [1] X is said to be *F-Menger* if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X each $\mathcal{U}_n$ contains a finite subfamily $\mathcal{V}_n$, $\mathcal{V}_n \subset \mathcal{U}_n$ such that $(\bigcup \mathcal{V}_n : n \in \mathbb{N})$ is an F-cover of X i.e. for every $x \in X$, $\{n \in \mathbb{N} : x \in \bigcup \mathcal{V}_n\} \in F$ (It should be noted that the definition of F-Menger property was actually presented in [1] for F where $F \subset SF$, a family of semifilters). It is easy to observe that I-Hurewicz property and F-Menger property are exactly same (this was pointed out by Prof Boaz Tsaban in a personal communication).

For the next result we further recall that a space X is *Δ-Lindeloef* if each Δ-cover of X contains a countable Δ-cover where Δ is same as defined before.

Theorem 2.1. *If an ω-Lindeloef space X satisfies $S_{fin}(\Omega, I-\mathcal{O}^{gp})$ then X has the I-Hurewicz property provided that I is admissible or free and such that $S_1(F(I), F(I))$ holds.*

Open Problem: Though the above result (Theorem 2.1) is true for $I = I_{fin}$ and $I = I_d$ (Note 2.2) without any additional assumption (in fact the assumption trivially holds for $I = I_{fin}$), it is not clear whether it is true for any arbitrary ideal I. In this context it seems a natural question as to whether the above result can be proved for any arbitrary ideal I without any additional assumption or can be proved under some other condition (preferably weaker).

Theorem 2.2. *For a space X the following are equivalent:*

(1) X satisfies $\alpha_2(I-\Gamma, \Gamma)$;

(2) X satisfies $\alpha_3(I-\Gamma, \Gamma)$;

(3) X satisfies $\alpha_4(I-\Gamma, \Gamma)$;

(4) X satisfies $S_1(I-\Gamma, \Gamma)$;

(5) ONE has no winning strategy in the game $G_1(I-\Gamma, \Gamma)$ on X.

Recall [12] that a uniform space $(X, \mathbb{U})$ is called *uniformly Hurewicz* (also called *Hurewicz bounded* [13]) if for each sequence $(U_n : n \in \mathbb{N})$ of elements from $\mathbb{U}$, there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finite number of sets $U_n[A_n]$ (for $U \in \mathbb{U}$ and $A \subset X$ recall that $U[A] = \{x \in X : (x, y) \in U \text{ for some } y \in A\}$).

We shall call a uniform space $(X, \mathbb{U})$ to be *I-Hurewicz bounded* if for each sequence $(U_n : n \in \mathbb{N})$ of elements from $\mathbb{U}$, there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin U_n[A_n]\} \in I$.

It can be easily verified that uniformly continuous images and finite unions of I-Hurewicz bounded spaces are also I-Hurewicz bounded.

We now present a characterization of I-Hurewicz boundedness in terms of I-groupability which is modelled after Theorem 3.7 [5].

Theorem 2.3. *For a uniform space $(X, \mathbb{U})$ the following are equivalent.*

(1) X is I-Hurewicz bounded.

(2) For each sequence $(U_n : n \in \mathbb{N}) \subset \mathbb{U}$ *there is a sequence* $(B_n : n \in \mathbb{N})$ *of finite subsets of* X *such that* $\{U_n[B_n] : n \in \mathbb{N}\}$ *is an* I*-groupable cover of* X*.*

Theorem 2.4. *(cf. Theorem 3.8 [5]) Every subspace of an* I*-Hurewicz bounded uniform space* $(X, \mathbb{U})$ *is also* I*-Hurewicz bounded.*

Theorem 2.5. *If an* I*-Hurewicz bounded uniform space* $(X, \mathbb{U}_X)$ *is dense in a uniform space* $(Y, \mathbb{U})$ *then* Y *is also* I*-Hurewicz bounded provided* I *is a free or an admissible ideal.*

In line of Corollary 3.10 [5] here we can have

Corollary 2.1. *A uniform space* $(X, \mathbb{U})$ *is* I*-Hurewicz bounded if and only if its completion* $\tilde{X}$ *is so.*

Theorem 2.6. *The product* $Z = (X \times Y, \mathbb{U}_X \times \mathbb{U}_Y)$ *is* I*-Hurewicz bounded if both* $(X, \mathbb{U}_X)$ *and* $(Y, \mathbb{U}_Y)$ *are* I*-Hurewicz bounded.*

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