Variations on selective separability

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IVth Workshop on Coverings Selections and Games in Topology

Selective separability

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Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are collections of subsets of an infinite set X.
$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> *For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.*

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> *For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$.*

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Let $\mathcal{D}$ denote the family of dense subspaces of a topological space X, the selection principles
$S_{fin}(\mathcal{D}, \mathcal{D})$ is called selective separability $=$ M-separability
$S_1(\mathcal{D}, \mathcal{D}) = R$-separability

θ-density Velichko 1966

Let $x \in X$ and $A \subset X$

- x is a *θ-cluster* point of A if the closure of each neighbourhood U of x meets A. $Cl_\theta(A)$ is the set of all θ-cluster points of A,
- A is *θ-closed* if $Cl_\theta(A) = A$,
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If X contains a countable θ-dense subset, the X is said to be *θ-separable*. $\mathcal{D}_\theta$ denote the family of all θ-dense sets in a space X. A space X is said to be:

- *selectively θ-separable* or M_θ*-separable* if X satisfies $\mathsf{S}_{fin}(\mathcal{D}_\theta, \mathcal{D}_\theta)$;
- *strongly selectively θ-separable* or R_θ*-separable* if X satisfies $\mathsf{S}_1(\mathcal{D}_\theta, \mathcal{D}_\theta)$;
- GN_θ*-separable* if X satisfies $\mathsf{S}_1(\mathcal{D}_\theta, \mathcal{D}_\theta^{gp})$, where $A \in \mathcal{D}_\theta$ is in $\mathcal{D}_\theta^{gp}$ if it can be written as a union of finite sets B_n, $n \in \mathbb{N}$, such that $\forall U$ open , $\overline{U} \cap B_n \neq \emptyset$ for all but finitely many n.

Since $\mathcal{D} \subset \mathcal{D}_\theta$, we have

$$\begin{array}{ccc} \mathsf{S}_{fin}(\mathcal{D}_\theta, \mathcal{D}) & \Rightarrow & \mathsf{S}_{fin}(\mathcal{D}, \mathcal{D}) \\ \Downarrow & & \Downarrow \\ \mathsf{S}_{fin}(\mathcal{D}_\theta, \mathcal{D}_\theta) & \Rightarrow & \mathsf{S}_{fin}(\mathcal{D}, \mathcal{D}_\theta) \end{array}$$

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Examples

- Let $\mathbb{R}$ endowed with the open-minus-countable topology ϑ: $U \in \vartheta$ if $U = V \setminus C$, where V is open in the usual topology on $\mathbb{R}$, and C is a countable subset of $\mathbb{R}$. It satisfies $\mathsf{S}_{fin}(\mathcal{D}_\theta, \mathcal{D}_\theta)$ and does not satisfy neither $\mathsf{S}_{fin}(\mathcal{D}_\theta, \mathcal{D})$ nor $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.

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- Let $X = \mathbb{R} \cup \{p\}$, where $p \notin \mathbb{R}$ endowed with topology $\mathcal{T}$: a set $U \in \mathcal{T}$ if either $U = V \setminus C$ with V open in the usual topology on $\mathbb{R}$ and C countable in $\mathbb{R}$, or $p \in U$ and $X \setminus U$ is countable. It satisfies $S_{fin}(\mathcal{D}, \mathcal{D}_\theta)$ but not neither $S_{fin}(\mathcal{D}, \mathcal{D})$ nor $S_{fin}(\mathcal{D}_\theta, \mathcal{D})$.

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- $\mathcal{V}$ is called a $\overline{\pi}$-*base* for X if for any open set $U \subset X$ there is $V \in \mathcal{V}$ such that $\overline{V} \subset U$
- $\mathcal{V}$ is called a π_θ-*base* for X if for any open set $U \subset X$ there is $V \in \mathcal{V}$ such that $V \subset \overline{U}$

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Proposition

- If a space X has a countable π_θ-base, then it satisfies $\mathsf{S}_1(\mathcal{D}_\theta, \mathcal{D}_\theta)$.
- If a space X has a countable $\overline{\pi}$-base, then it satisfies $\mathsf{S}_1(\mathcal{D}_\theta, \mathcal{D})$.
- The product $X \times Y$ of a space X satisfying $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D}_\theta)$ and a space Y having a countable π_θ-base satisfies also $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D}_\theta)$.

X has countable *θ-fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in \omega)$ of subsets of X such that $x \in Cl_\theta(A_n)$ for all $n \in \omega$, there are finite sets $F_n \subset A_n$, $n \in \omega$, such that $x \in Cl_\theta(\bigcup_{n\in\omega} F_n)$.

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Proposition

If a θ-separable space X has countable θ-fan tightness, then X is selectively M_θ-separable.

The Alexandroff Duplicate

Let $f : X \to Y$ is said to be θ-*continuous* if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(\overline{U}) \subset \overline{V}$.

- If $f : X \to Y$ is a closed irreducible mapping and U is an open subset of X, then $f(\overline{U}) = \overline{f^{\#}(U)}$.
- If $f : X \to Y$ is closed irreducible, and D is θ-dense subset of Y, then $f^{\leftarrow}(D)$ is θ-dense in X.

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The M_θ-separability is preserved by closed irreducible θ-continuous mappings.

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The M_θ-separability is preserved by closed irreducible θ-continuous mappings.

Corollary

The Alexandroff duplicate $\mathrm{AD}(X)$ of a space X is M_θ-separable if and only if X is M_θ-separable

The unexpected result that separable Fréchet-Urysohn spaces are M-separable was shown in Barman-Dow.

X is *θ-Fréchet-Urysohn* if for each $A \subset X$ and each $x \in \mathrm{Cl}_\theta(A)$ there is a sequence $(a_n)_{n\in\omega}$ in A which θ-converges to x, i.e. for each open set U containing x there is $n_0 \in \omega$ such that $a_n \in \overline{U}$ for all $n \geq n_0$.

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Proposition

Let X be a Urysohn space such that for each $A \subset X$ and each $x \in X$, $x \in \mathrm{Cl}_\theta(A \setminus \{x\})$. If X is θ-separable and θ-Fréchet-Urysohn, then X is M_θ-separable.

Game G_{pp} (*resp.* G_{pp}^{gp})

Two players, ONE and TWO, play a countably infinite game on a space X. In the n-th round ONE chooses a nonempty open set U_n and TWO responds by choosing a point $x_n \in \overline{U_n}$. ONE wins a play $(U_1, x_1; U_2, x_2; \cdots ; U_n, x_n; \cdots)$ if the set $\{x_n : n \in \omega\}$ is θ-dense (*resp. θ-groupable dense*) in X; otherwise TWO wins.

Point-picking Games

Game G_{pp} (*resp.* G_{pp}^{gp})

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Game $G_1(\mathcal{A}; \mathcal{B})$

Two players, ONE and TWO, play an inning per positive integer. In the n-th inning ONE chooses a set O_n from $\mathcal{A}$, and TWO responds by choosing an element $T_n \in O_n$. The play $(O_1; T_1; \ldots; O_n; T_n; \ldots)$ is won by TWO if $\{T_n : n \in \omega\}$ is a member of $\mathcal{B}$; otherwise, ONE wins.

Theorem

The following are equivalent for a space X:

(1) ONE has a winning strategy in the game $\mathcal{G}_1(\mathcal{D}_\theta, \mathcal{D}_\theta)$ on X;

(2) TWO has a winning strategy in the game G_{pp} on X.

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Proposition

If ONE has a winning strategy in the game G_{pp} on a space X, then TWO has a winning strategy in $\mathcal{G}_1(\mathcal{D}_\theta, \mathcal{D}_\theta)$.

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The following are equivalent for a space X:

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Proposition

If ONE has a winning strategy in the game G_{pp} on a space X, then TWO has a winning strategy in $\mathcal{G}_1(\mathcal{D}_\theta, \mathcal{D}_\theta)$.

Theorem

For a space X the following assertions are equivalent:

(1) ONE has a winning strategy in the game $\mathcal{G}_1(\mathcal{D}_\theta, \mathcal{D}_\theta^{gp})$ on X;

(2) TWO has a winning strategy in the game G_{pp}^{gp} on X.

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