Modifications of Sequence Selection Principles

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For simplicity
all considered topological spaces
are supposed
to be perfectly normal

Sequence Selection Property

A.V. Arkhangel'skiĭ [1972] introduced properties (α_i), $i = 1, 2, 3, 4$ of a topological space X which are actually sequence selection properties:

X has the property (α_i) if for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x, there exists a sequence $\{y_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} y_m = x$ and

(α_1) $\{x_{n,m} : m \in \omega\} \subseteq^* \{y_m : m \in \omega\}$ for each n,

(α_2) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for each n,

(α_3) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for infinitely many n,

(α_4) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\} \neq \emptyset$ for infinitely many n.

It is easy to see that

$$(\alpha_1) \to (\alpha_2) \to (\alpha_3) \to (\alpha_4).$$

Theorem (D. Fremlin [1994])

The following are equivalent:

(1) X *is an* s_1*-space,*

(2) *whenever* $\lim_{n\to\infty} f_{m,n} = 0$ *for each* m, $f_{m,n} \in \mathrm{C}_p(X)$, *then there exists an increasing sequence* $\{n_m\}_{m=0}^{\infty}$ *such that* $\lim_{m\to\infty} f_{m,n_m} = 0$.

X is a s_1-space if for any set $A \subseteq \mathrm{C}_p(X)$

$$\mathrm{sc}(\mathrm{sc}(A)) = \mathrm{sc}(A),$$

where sc(A) is the sequential closure of A:

$$\mathrm{sc}(A) = \{f \in {}^X\mathbb{R} : (\exists \{f_n\}_{n=0}^{\infty} \subseteq A) \lim_{n\to\infty} f_n = f\}.$$

M. Scheepers [1997] introduced the condition (2) as a sequence selection property of $C_p(X)$, shortly SSP.

He notes that

$$C_p(X) \text{ has SSP} \equiv C_p(X) \text{ satisfies } (\alpha_2)$$

F. Gerlits and Z. Nagy [1982] and independently M. Scheepers [1998] proved that

$$(\alpha_2) \equiv (\alpha_4) \text{ for } C_p(X).$$

A. Dow proved that

$$(\alpha_1) \equiv (\alpha_2) \text{ in Laver model.}$$

There exist models of **ZFC** in which

$$(\alpha_2) \nrightarrow (\alpha_1) \text{ even for } C_p(X).$$

A sequence $\{f_n\}_{n=0}^{\infty}$ of real-valued functions converges to f on X quasi-normally, written $f_n \xrightarrow{\text{QN}} f$ on X, if there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive reals (the control sequence), $\lim_{n\to\infty} \varepsilon_n = 0$ and such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\, |f_n(x) - f(x)| < \varepsilon_n.$$

$f_n \xrightarrow{\text{QN}} f$ on X if and only if $X = \bigcup_k X_k$, $X_k \subseteq X_{k+1}$, $f_n \rightrightarrows f$ on X_k for each k. If f_n are continuous, then X_k are closed.

L. Bukovský, I. Recław and M. Repický [1991]:
X is a QN-space if for any sequence $f_n \in \mathrm{C}_p(X)$, $f_n \to 0$ on X also $f_n \xrightarrow{\text{QN}} 0$ on X.
X is a wQN-space if for any sequence $f_n \in \mathrm{C}_p(X)$, $f_n \to 0$ on X, there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{QN}} 0$ on X.

M. Scheepers [1999]: Any s_1-space is a wQN-space.
D. Fremlin [2002, unpublished]: Any wQN-space is an s_1-space.

Theorem

If X is a perfectly normal topological space, then the following are equivalent:

(1) *X is a wQN-space.*

(2) *$\mathrm{C}_p(X)$ has SSP.*

(3) *$\mathrm{C}_p(X)$ posseses the property (α_2).*

M. Scheepers [1999]: If $\mathrm{C}_p(X)$ has the property (α_1), then X is a QN-space.
M. Sakai [2007] and independently L. Bukovský and J. Haleš [2007]: If X is a QN-space, then $\mathrm{C}_p(X)$ posseses the property (α_1).

Theorem

X is a QN-space if and only if $\mathrm{C}_p(X)$ posseses the property (α_1).

X is an mQN-space if for any sequence $f_n \in \mathrm{C}_p(X)$, $f_n \to 0$ on X, and $f_{n+1} \leq f_n$, also $f_n \xrightarrow{\mathrm{QN}} 0$ on X.
X has Hurewicz property if for any sequence $f_n \in \mathrm{C}_p(X)$ the family of sequences of reals $\{\{f_n(x)\}_{n=0}^{\infty} : x \in X\} \subseteq {}^{\omega}\mathbb{R}$ is eventually bounded.
L. Bukovský, I. Recław and M. Repický [2001] for metric space,
L. Bukovský and J. Haleš [2003] for any topological space proved

Theorem

The following are equivalent:

(1) *X is an mQN-space.*

(2) *X has the Hurewicz property.*

(3) *Whenever $\lim_{n\to\infty} f_{m,n} = 0$ for each m, $f_{m,n} \in \mathrm{C}_p(X)$ and $f_{m,n} \geq f_{m,n+1}$ for any m, n, then there exists an increasing sequence $\{n_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} f_{m,n_m} = 0$.*

The condition (3) is a sequence selection property.

The First Modifications

$\mathcal{U} \subseteq \mathcal{P}(X)$ is a cover of X, if $\bigcup \mathcal{U} = X$ and $X \notin \mathcal{U}$.
A cover $\mathcal{U}$ is open, closed, Borel, if each set $A \in \mathcal{U}$ is open, closed, Borel, respectively.
A cover $\mathcal{U}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{U}$. Γ is the family of all open γ-covers of X.
A cover $\mathcal{U}$ is an ω-cover if every finite subset of X lies in some member of $\mathcal{U}$.
A cover $\mathcal{U}$ is a refinement of a cover $\mathcal{V}$, if for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ such that $U \subseteq V$.
An open γ-cover is shrinkable if there exists a closed γ-cover which is its refinement. Γ^{sh} is the family of all shrinkable open γ-covers.

Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of a topological space X.
M. Scheepers [1996]: X has the property $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ if for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of covers of the family $\mathcal{A}$, there exist $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\}$ is a cover of X belonging to $\mathcal{B}$.

M. Scheepers [1999]: An $\mathrm{S}_1(\Gamma,\Gamma)$-space is a wQN-space.

Scheepers Conjecture: $\mathrm{wQN} = \mathrm{S}_1(\Gamma,\Gamma)$

L. Bukovský and J. Haleš [2007]: $\mathrm{wQN} = \mathrm{S}_1(\Gamma^{sh},\Gamma)$.

By L. Bukovský and J. Haleš [2007] and M. Sakai [2007]:

$\mathrm{QN} \subseteq \mathrm{S}_1(\Gamma,\Gamma)$.

By A. Dow's result

ZFC+$\mathrm{wQN} = \mathrm{S}_1(\Gamma,\Gamma) = \mathrm{QN}$ is consistent.

W. Just, A.W. Miller, M. Scheepers and Szeptycki [1996]:

ZFC+$\mathrm{S}_1(\Gamma,\Gamma) \neq \mathrm{QN}$ is consistent.

Open Problem: Consistency of $\mathrm{wQN} \neq \mathrm{S}_1(\Gamma,\Gamma)$.

Let Γ_{cl} be the family of all clopen γ-covers.
Based on the work of J. Haleš [2005] we have obtained

Theorem (M. Sakai [2007], L. Bukovský and J. Haleš [2007])

X is a wQN-space if and only if $\mathrm{Ind}(X){=}0$ and X is an $\mathrm{S}_1(\Gamma_{cl},\Gamma)$ space.

Thus the existence of a large zero dimensional space which has the property $\mathrm{S}_1(\Gamma_{cl},\Gamma)$ and does not have the property $\mathrm{S}_1(\Gamma,\Gamma)$ solves the Scheepers Conjecture in a negative way.

If in the definitions of wQN-space, SSP property we allow lower semicontinuous and upper semicontinuous functions instead of continuous, we obtain the notions of

wQN_*-space, wQN^*-space, SSP_* and SSP^* property,

respectively.

L. Bukovský [2008] with a help of M. Sakai and B. Tsaban obtained:

$$\mathrm{SSP}^* = \mathrm{wQN}^* = \mathrm{S}_1(\Gamma, \Gamma),$$

$$\mathrm{SSP}_* = \mathrm{wQN}_* = \mathrm{QN}.$$

Thus

$$\mathrm{wQN}_* \subseteq \mathrm{wQN}^* \subseteq \mathrm{wQN}$$

and

$$\mathrm{SSP}_* \subseteq \mathrm{SSP}^* \subseteq \mathrm{SSP}.$$

The Second Modification

Theorem (L. Bukovský, I. Recław and M. Repický [2001])

The following are equivalent:

(a) *For every Borel measurable function $f : X \longrightarrow {}^{\omega}\omega$ the range $f(X)$ is eventually bounded.*

(b) *X is a QN-space and every Borel measurable function from X into ${}^{\omega}\omega$ is a discrete limit of continuous functions.*

Theorem (B. Tsaban and L. Zdomskyy [2012], preprint 2006)

If X is a QN-space, then any Borel measurable function from X into ${}^{\omega}\omega$ is eventually bounded.

Our aim was to show, using a modified sequence selection principle:

If X is a QN-space, then every Borel measurable function from X into ${}^{\omega}\omega$ is a discrete limit of continuous functions.

The letters P, Q, D denote the pointwise convergence, quasi-normal convergence and discrete convergence of a sequence of real functions, respectively. Let A, B be some of P, Q, D. If $\mathcal{A}$, $\mathcal{B}$ are sets of functions from a topological space X into $\mathbb{R}$, then the selection principle AB$(\mathcal{A}, \mathcal{B})$ means the following:

for any system of functions $f^n_m \in \mathrm{C}_p(X)$, $f_n \in \mathcal{A}$, $f \in \mathcal{B}$, $n, m \in \omega$

such that: $f_n \xrightarrow{\mathrm{A}} f$ on X and $f^n_m \xrightarrow{\mathrm{A}} f_n$ on X for each n,

there exists an unbounded $\{m_n\}_{n=0}^{\infty}$ such that $f^n_{m_n} \xrightarrow{\mathrm{B}} f$ on X.

PP$(\mathrm{C}_p(X), \{0\})$ is the sequence selection principle SSP.

If we ask the existence of two unbounded sequences $\{k_n\}_{n=0}^{\infty}$ and $\{m_n\}_{n=0}^{\infty}$ such that $f^{k_n}_{m_n} \xrightarrow{\mathrm{B}} f$ on X, we obtain the weak sequence selection principle wAB$(\mathcal{A}, \mathcal{B})$.

$$
\begin{array}{ccccc}
 & & & & \mathrm{DD}(\mathcal{A},\mathcal{B}) \\
 & & & & \downarrow \\
\mathrm{PQ}(\mathcal{A},\mathcal{B}) & \longrightarrow & \mathrm{QQ}(\mathcal{A},\mathcal{B}) & \longrightarrow & \mathrm{DQ}(\mathcal{A},\mathcal{B}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PP}(\mathcal{A},\mathcal{B}) & \longrightarrow & \mathrm{QP}(\mathcal{A},\mathcal{B}) & \longrightarrow & \mathrm{DP}(\mathcal{A},\mathcal{B})
\end{array}
$$

$$\mathrm{AB}({}^X\mathbb{R}, {}^X\mathbb{R}) \to \mathrm{AB}({}^X\mathbb{R}, \{0\}) \to \mathrm{AB}(\mathrm{C}_p(X), \{0\}) \to \mathrm{AB}(\{0\}, \{0\})$$

Similarly for wAB.

$\mathrm{wAB}(\{0\},\{0\}) \equiv \mathrm{AB}(\{0\},\{0\})$ for any A,B, $\mathrm{B} \neq \mathrm{D}$

Theorem (Mostly J. Šupina [2012])

The following are equivalent:

X *is a* QN*-space*
X *satisfies* $\mathrm{DD}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{QQ}({}^X\mathbb{R}, {}^X\mathbb{R}\})$,
X *satisfies* $\mathrm{PQ}({}^X\mathbb{R}, {}^X\mathbb{R}\})$,
X *satisfies* $\mathrm{DD}({}^X\mathbb{R}, {}^X\mathbb{R}\})$,
X *satisfies* $\mathrm{PQ}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{wDD}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{wQQ}({}^X\mathbb{R}, {}^X\mathbb{R}\})$,
X *satisfies* $\mathrm{wPQ}({}^X\mathbb{R}, {}^X\mathbb{R}\})$,
X *satisfies* $\mathrm{wDD}({}^X\mathbb{R}, {}^X\mathbb{R}\})$.

Theorem (Mostly J. Šupina [2012])

The following are equivalent:

X *is a* wQN*-space*
X *satisfies* $\mathrm{QP}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{PP}(\{0\}, \{0\})$,
X *satisfies* $\mathrm{PQ}(\{0\}, \{0\})$,
X *satisfies* $\mathrm{PP}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{QQ}(\mathrm{C}_p(X), \{0\})$,
X *satisfies* $\mathrm{QP}(\{0\}, \{0\})$,
X *satisfies* $\mathrm{QQ}(\{0\}, \{0\})$.

Theorem (I. Recław [1997])

A topological space X satisfying $\mathrm{DP}({}^X\mathbb{R}, \{0\})$ is a σ-space.

Proof. Let $F = \bigcup_n F_n$ be an F_σ set, $F_n \subseteq F_{n+1}$ being closed. The open sets $X \setminus F_n$ may be written as $X \setminus F_n = \bigcup_m F_{n,m}$, where $F_{n,m} \subseteq F_{n,m+1}$ are closed. There exist continuous functions $f_{n,m}$ such that $f_{n,m}(x) = 1$ for $x \in F_n$ and $f_{n,m}(x) = 0$ for $x \in F_{n,m}$. Then $f_{n,m} \xrightarrow{\mathrm{D}} \chi_{F_n}$ and $\chi_{F_n} \xrightarrow{\mathrm{D}} \chi_F$. By $\mathrm{DP}({}^X\mathbb{R}, \{0\})$ there exists an increasing $\beta \in {}^\omega\omega$ such that $f_{n,\beta(n)} \to \chi_F$. Thus F is a G_δ set.

q.e.d.

Since

$$\mathrm{DD}({}^X\mathbb{R}, {}^X\mathbb{R}) \to \mathrm{DP}({}^X\mathbb{R}, \{0\})$$

we obtain

Every QN-space is a σ-space

and

Borel measurable function from a QN-space is Δ^0_2-measurable.

Theorem (L. Bukovský and J. Šupina [2012])

If $\mathrm{Ind}(X) = 0$, *then for any* Δ^0_2*-measurable function* $f : X \longrightarrow \langle 0, 1 \rangle$ *there exist* Δ^0_2*-measurable functions* $f_n : X \longrightarrow \langle 0, 1 \rangle$ *and continuous functions* $f_{n,m} : X \longrightarrow \langle 0, 1 \rangle$ *such that* $f_n \rightrightarrows f$ *on* X *and* $f_{n,m} \to f_n$ *on* X *for each* n.

Corollary (L. Bukovský and J. Šupina [2012])

If a topological space X *possesses the property* $\mathrm{QQ}({}^X\mathbb{R}, {}^X\mathbb{R})$ *then every Borel measurable function is a discrete limit of continuous functions.*

Proof of Tsaban – Zdomskyy's Theorem:

If $f_n \xrightarrow{\mathrm{D}} f$ on X then $X = \bigcup_m X_m$, each X_m closed, and $f|X_m = f_n|X_m$ for sufficiently large n. A continuous image of a QN-space into ${}^\omega\omega$ is eventually bounded.

q.e.d.

Theorem (J. Šupina [2012])

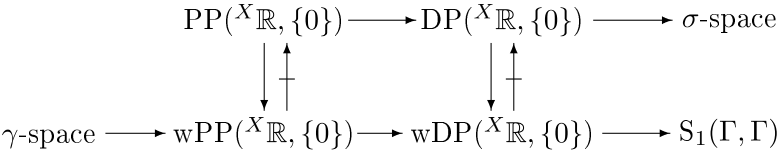
a) *A topological space satisfying* $\mathrm{DP}({}^X\mathbb{R}, \{0\})$ *is a* σ*-space.*

b) *A topological space satisfying* $\mathrm{wDP}({}^X\mathbb{R}, \{0\})$ *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space.*

c) *Any* γ*-space satisfies principle* $\mathrm{wPQ}({}^X\mathbb{R}, \{0\})$.

Theorem (F. Galvin and A. Miller [1984])

If $\mathfrak{p} = \mathfrak{c}$ *then there exists a* γ*-space that is not a* σ*-space.*



Note that $\mathrm{wPP}(\mathrm{C}_p(X), \{0\}) \longrightarrow \mathrm{PP}(\mathrm{C}_p(X), \{0\})$.

Thank You!

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