$S_1(\Gamma, \Gamma)$ ***vs*** $S_1(\Gamma_{clopen}, \Gamma_{clopen})$

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Mikhail “Misha”Matveev passed away unexpectedly on May 17, 2011

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Outline

Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set.

M. Scheepers:

$S_1(\mathcal{A}, \mathcal{B})$: for each sequence $(A_n : n \in \omega)$ of elements of $\mathcal{A}$ there is a sequence $(a_n : n \in \omega)$ such that for each $n \in \omega$, $a_n \in A_n$ and $\{a_n : n \in \omega\}$ belongs to $\mathcal{B}$.

A family $\mathcal{U}$ of subsets of a set X is called a *point-cofinite* cover (even γ-cover) if $\mathcal{U}$ is infinite and every $x \in X$ is contained in all but finitely many elements of $\mathcal{U}$.

Γ: the family of all open point-cofinite covers.

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Γ_{cl} = the familiy of all point-cofinite covers by clopen sets.

All spaces considered are Tychonoff.

$C_p(X)$= the space of all real-valued continuous functions on X with the topology of pointwise convergence.

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Question [Scheepers1999]

$S_1(\Gamma_0, \Gamma_0)$ for $C_p(X) \Rightarrow^? S_1(\Gamma, \Gamma)$ for X, for a perfectly normal space X

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Theorem [Sakai2007, (see also Bukovský- J.Haleš 2007)]
For a normal space X,

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Question 1 [Scheepers]:
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Question 2 [M. Sakai]:
- Is there a Tychonoff space which is $S_1(\Gamma_{cl}, \Gamma_{cl})$ and is not $S_1(\Gamma, \Gamma)$?

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We answer Question 2 in the negative, under CH.

Our example does not answer Question 1, because it is not perfectly normal (since it contains a copy of ω_1 as a closed subspace).

It also distinguishes the property $\mathrm{S}_1(\Gamma_{cl}, \Gamma_{cl})$ for X from the property $S_1(\Gamma_0, \Gamma_0)$ for $C_p(X)$ in the class of normal zero-dimensional non strongly zero-dimensional spaces.

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Example: [Sakai2010]
Isbell-Mrówka space $X = \Psi(\mathcal{E})$, where $\mathcal{E}$ is an infinite maximal almost disjoint family consisting of infinite subsets of ω.

X is a
- strongly zero-dimensional
- perfect
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space satisfying $S_1(\Gamma_{cl}, \Gamma_{cl})$ but not $S_1(\Gamma, \Gamma)$.

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Example (CH) A space having the $S_1(\Gamma_{cl}, \Gamma_{cl})$ property and not having the $S_1(\Gamma, \Gamma)$ property.

Definition A space X is *clopen trivial* if every clopen set in it is countable or co-countable.

Lemma If X is an uncountable set, and $\mathcal{U}$ is a point-cofinite cover of X such that $\mathcal{U}$ consists only of countable and co-countable sets, then there is an infinitely countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is a point-cofinite cover of X and $\mathcal{U}_0$ consists only of co-countable sets.
Proof:

- We may assume that $\mathcal{U}$ is a *countable*.
- Then $\mathcal{U}$ contains only finitely many countable elements.
- Thus, removing from $\mathcal{U}$ the finitely many countable elements, we obtain a countable point-cofinite cover consisting of co-countable sets.

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Proposition 1 Every clopen-trivial space X is $S_1(\Gamma_{cl}, \Gamma_{cl})$.

Proof:

- X is countable $\Rightarrow$ X is $S_1(\Omega, \Gamma) \Rightarrow S_1(\Gamma_{cl}, \Gamma_{cl})$.
- Assume that X is uncountable.
- Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of countable point-cofinite cover consisting of co-countable clopen sets.
- Then the set $S = \bigcup\{X \setminus U : U \in \mathcal{U}_n, n \in \omega\}$ is countable.
- Enumerate $S = \{x_n : n \in \omega\}$.
- For every $n \in \omega$ we may pick an element $U_n \in \mathcal{U}_n$ such that $\{x_m : m \leq n\} \subset U_n$.
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Proposition 1 Every clopen-trivial space X is $S_1(\Gamma_{cl}, \Gamma_{cl})$.
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- Q= the set of all rational numbers in the unit interval I.
- For $x, y \in I$, set $x \approx y$ if $|x - y| \in Q$.
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We modified the previous construction.

- (CH) I can be partitioned in ω_1-many countable, dense subsets Q_α, $\alpha \in \omega_1$.
- Let $X = \bigcup_{\gamma<\omega_1}(\{\gamma\} \times S_\gamma)$, where $S_\gamma = \bigcup\{Q_\alpha : \alpha \leq \gamma\}$.
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X has the properties of the original Dowker's example.

X is not $S_1(\Gamma, \Gamma)$:
because the restricion $\pi_2|X$ to X of the projection π_2 of the product $\omega_1 \times I$ to the second factor, is a continuous mapping onto I, which does not satisfy the property $S_1(\Gamma, \Gamma)$.

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X is $S_1(\Gamma_{cl}, \Gamma_{cl})$: We will proof that X is clopen-trivial.

$\forall x \in I$, $\forall \epsilon \in \mathbb{R}$, $\forall \alpha < \omega_1$,

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- Any two 1-point wide tails can not be separated by clopen sets.

Let $\mathcal{B}$ be a countable base of I and $U \subset X$ be a clopen, but not countable set.
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<u>*Claim*2 :</u> Let $\{C_1, C_2\}$ be a partition of X into clopen sets and $x \in I$. Then only one of the sets $C_1 \cap (\omega_1 \times \{x\})$, $C_2 \cap (\omega_1 \times \{x\})$ contains a tail.

<u>*Claim*3 :</u> Let $\{C_1, C_2\}$ be a partition of X into clopen sets. If $I_j = \{x \in I : C_j \cap (\omega_1 \times \{x\})$ contains a tail$\}$, where $j \in \{1, 2\}$, then $\{I_1, I_2\}$ is a partition of I into open sets. In particular, since I is connected, one of I_j is empty and then the other concides with I.

Proof:

- Fix $j \in \{1, 2\}$.
- $\forall\ x \in I$ s.t. $C_j \cap (\omega_1 \times \{x\})$ contains a tail $\Rightarrow^{Claim1}$ $\exists\ \epsilon_x > 0$ s.t. a $B(x, \epsilon_x)$-wide tail $\subset C_j$.
- Hence $\bigcup_{x \in I_j} B(x, \epsilon_x) \subset C_j$ and then I_j is open.

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<u>*Claim2 :*</u> Let $\{C_1, C_2\}$ be a partition of X into clopen sets and $x \in I$. Then only one of the sets $C_1 \cap (\omega_1 \times \{x\})$, $C_2 \cap (\omega_1 \times \{x\})$ contains a tail.

<u>*Claim3 :*</u> Let $\{C_1, C_2\}$ be a partition of X into clopen sets. If $I_j = \{x \in I : C_j \cap (\omega_1 \times \{x\})$ contains a tail$\}$, where $j \in \{1, 2\}$, then $\{I_1, I_2\}$ is a partition of I into open sets. In particular, since I is connected, one of I_j is empty and then the other concides with I.

Proof:

- Fix $j \in \{1, 2\}$.
- $\forall\, x \in I$ s.t. $C_j \cap (\omega_1 \times \{x\})$ contains a tail $\Rightarrow^{Claim1}$ $\exists\, \epsilon_x > 0$ s.t. a $B(x, \epsilon_x)$-wide tail $\subset C_j$.
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<u>*Claim4 :*</u> If $K \subset X$ is a clopen set such that

$$\{x \in I : K \cap (\omega_1 \times \{x\}) \text{ contains a tail}\} = I,$$

then K contains a I-wide tail.

Proof:

- Let $\tilde{\mathcal{B}} = \{B \in \mathcal{B} : \exists \alpha_B \in \omega_1 \text{ s.t. } ([\alpha_B, \omega_1) \times B) \cap X \subset K\}$.
- $\tilde{\mathcal{B}}$ covers I.
- Put $\alpha^* = \sup\{\alpha_B : B \in \tilde{\mathcal{B}}\}$.
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$\underline{Claim 5:}$ For any partition of X into two clopen sets, one of them is bounded or contains a I-wide tail.

Thus U is co-countable.

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Some consequences

Corollary (CH) There exists a zero-dimensional normal non strongly zero-dimensional space X such that X is $S_1(\Gamma_{cl}, \Gamma_{cl})$, but $C_p(X)$ is not $S_1(\Gamma_0, \Gamma_0)$.

Definition [D. Shakmatov1990, A.V. Arhangelskii1992]
X is a α_2-*space* if for every $x \in X$ and every sequence $\{S_n : n \in \omega\}$ of non-trivial sequences converging to x, there is a sequence S converging to x such that $S_n \cap S$ is infinite for all n.

[Scheepers1998, Saka2007]: A space of the form
$C_p(Y)$ is an α_2-space $\Leftrightarrow$ $C_p(Y)$ is $S_1(\Gamma_0, \Gamma_0)$
Proof. of Corollary:

- X= the space of Example
- $\pi_2(X) = I \Rightarrow C_p(X)$ contains a closed subspace homeomorphic to $C_p(I)$.
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Some consequences

Corollary (CH) There exists a zero-dimensional space X such that $C_p(X,2)$ is an α_2-space, but $C_p(X)$ is not an α_2-space.

For zero-dimensional spaces X,

$C_p(X,2)$ is an α_2-space $\Leftrightarrow$ X has property $S_1(\Gamma_{cl},\Gamma_{cl})$

(see also [BonanzingaCammarotoMatveev2010])

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Grazie!!!