When is a Pixley-Roy hyperspace SS$^+$?

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For a given space X, the Pixley-Roy hyperspace $PR(X)$ is the space of all non-empty finite subsets of X equipped with the topology generated by sets of the form $[A, U] = \{B \in PR(X) : A \subseteq B \subseteq U, A \in PR(X) \text{ and } U \text{ open in } X\}$.

Proposition 1. *Let X be a space.*
a) $PR(X)$ is a T_2 zero-dimensional space;
b) $PR(X)$ is separable if and only if X is countable (and hence in and only if $PR(X)$ is countable);
c) $PR(X)$ has countable π-weight if and only if X is countable and first countable.

A space X is *selectively separable* (briefly SS) if the selection principle $S_{fin}(\mathcal{D},\mathcal{D})$ hols, i.e. for every sequence $(D_n : n < \omega)$ of dense subspaces of X one can pick finite $F_n \subset D_n$, $n < \omega$, in such a way that $\bigcup\{F_n : n < \omega\}$ is dense in X.

A space X is *strategic selectively separable*, briefly SS$^+$, if player II has a winning strategy in the game $G_{fin}(\mathcal{D},\mathcal{D})$ played as follows: at the n-th inning player I choose a dense set D_n and player II responds by selecting a finite set $F_n \subseteq D_n$. Player II wins if the set $\bigcup\{F_n : n < \omega\}$ is dense in x.

$$Countable\ \pi - \mathrm{weight}\ \rightarrow\ SS^+\ \rightarrow\ selectively\ separable$$

$\beta\omega$ and 2^ω are obviously SS$^+$.

$C_p(2^\omega, 2)$ is a SS$^+$ countable topological group of uncountable π-weight.

2^{ω_1} is a compact separable space which is not selectively separable.

Fact 1. *Every SS$^+$ space is resolvable, i. e. it contains two disjoint dense sets.*

Fact 2. $[\mathfrak{d} = \mathfrak{c}]$ *There exists a selectively separable irresolvable space.*

A space X has **strategic fan tightness** at a point $x \in X$ if player II has a winning strategy in the following game:

at the n-th inning player I playes a set $A_n \subseteq X$ with $x \in \overline{A_n}$ and player II responds by selecting a finite set $F_n \subseteq A_n$. Player II wins if and only if $x \in \overline{\bigcup\{F_n : n < \omega\}}$.

By restricting the moves of player I in the above game to dense sets only, we get the weaker notion of **strategic dense fan tightness**.

Proposition 2. *A space is SS^+ if and only if it is a separable space of strategic dense fan tightness.*

Recall that a collection $\mathcal{P}$ of non-empty subsets of the space X is a π-network at a point $x \in X$ provided that every neighbourhood of x contains some element of $\mathcal{P}$.

A space X has **strategic fan tightness for finite sets** at $x \in X$ if player II has a winning strategy in the following game:

at the n-th inning player I choose a collection $\mathcal{P}_n \subseteq [X]^{<\omega}$ which is a π-network at x and player II responds by selectin a finite set $\mathcal{Q}_n \subseteq \mathcal{P}_n$. Player II wins if and only if the set $\bigcup\{\mathcal{Q}_n : n < \omega\}$ is a π-network at x.

Theorem 1. *For a space X the following assertions are equivalent:*
a) $PR(X)$ has strategic fan tightness;
b) $PR(X)$ has strategic dense fan tightness;
c) X^k has strategic fan tightness for finite sets for each integer k.

Corollary 1. *$PR(X)$ is SS^+ if and only if X is countable and X^k has strategic fan tightness for finite sets for each integer k.*

Theorem 2. *If X is σ-compact, then $C_p(X)$ has strategic fan tightness for finite sets.*

Corollary 2. *Let X be a σ-compact space and k an integer. If Y is a subspace of $C_p(X)$, then Y^k has strategic fan tightness for finite sets.*

Corollary 3. *If X is a σ-compact space and Y is a countable subspace of $C_p(x)$, then $PR(Y)$ is SS^+.*

Corollary 4. *$PR(C_p(2^\omega, 2))$ is a SS^+ Pixley-Roy hyperspace of uncountable π-weight.*

Theorem 3. *There is a selectively separable Pixley-Roy hyperspace with is not SS^+.*

Proof. Barman and Dow have shown that there is a space X such that the function space $C_p(X)$ has countable fan tightness, but there is a countable subspace $Y \subseteq C_p(X)$ which is not SS^+. By a theorem of Sakai,

$PR(Y)$ is selectively separable. However, the space $PR(Y)$ cannot be SS^+ because by Proposition 2 Y does not have strategic fan tightness and so, a fortiori, it does not have strategic fan tightness for finite sets in each finite power. $\square$