A TASTE OF SET THEORY: EXERCISE NUMBER 9

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All exercises below are taken from the booklet. Most of the solutions are very short. If needed, there are some hints in the booklet.

For $X \subseteq \mathbb{R}$, the *derived set* X' is the set of all accumulation (or: limit) points of X.

1. Let $X \subseteq \mathbb{R}$. Prove:

(1) $X \setminus X'$ countable.
(2) If X' is countable, then X is countable.
(3) X' is closed.
(4) If there are no isolated points in X, then $X \subseteq X'$.
(5) If X is closed, then $X' \subseteq X$.
(6) X is perfect if, and only if, $X = X'$.

Define, by transfinite recursion:

(1) $X^{(0)} = X$;
(2) $X^{(\alpha+1)} = (X^{(\alpha)})'$;
(3) For a limit ordinal α, $X^{(\alpha)} = \bigcap\{X^{(\beta)} : 0 < \beta < \alpha\}$.

2. Let $X \subseteq \mathbb{R}$. Prove:

(1) For each $\alpha > 0$, $X^{(\alpha)}$ is closed.
(2) For each $0 < \alpha < \beta$, $X^{(\beta)} \subseteq X^{(\alpha)}$.
(3) For each α, if $X^{(\alpha)} = X^{(\alpha+1)}$, then for each $\beta > \alpha$, $X^{(\beta)} = X^{(\alpha)}$.
(4) If $X^{(\alpha+1)} \subsetneq X^{(\alpha)}$, then for each $\beta < \alpha$, $X^{(\beta+1)} \subsetneq X^{(\beta)}$.

3. Assume that $\langle F_\beta : \beta < \alpha\rangle$ is a strictly decreasing sequence of closed subsets of $\mathbb{R}$ (for each $\gamma < \delta$, $F_\delta \subsetneq F_\gamma$). Prove that $\alpha < \aleph_1$.

Good luck!

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