A TASTE OF SET THEORY: EXERCISE NUMBER 8

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1. Assume that $\lambda < \mathrm{cf}(\kappa)$ and for each $\alpha < \lambda$, $|X_\alpha| < \kappa$. Show that $|\bigcup_{\alpha<\lambda} X_\alpha| < \kappa$ (in particular, $\bigcup_{\alpha<\lambda} X_\alpha \neq \kappa$).
Hint. Define $f : \lambda \to \kappa$ by $f(\alpha) = |X_\alpha|$. f is bounded.

2. Prove: For each κ, $\mathrm{cf}(2^\kappa) > \kappa$.
Hint. Zermelo-König Lemma.

3. Supply short proofs to each of the following.

(1) Assume that $\lambda < \mathrm{cf}(\kappa)$. Show:
   (a) ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$.
   (b) $\kappa^\lambda \leq \kappa \cdot \sup\{\mu^\lambda : \mu < \kappa\}$ (Use (a)).
   (c) $\kappa^\lambda \leq \kappa \cdot \sup\{2^\mu : \mu < \kappa\}$ (Use (b)).

(2) Assume GCH. Show that for all infinite cardinals κ and λ:

$$\kappa^\lambda = \begin{cases} \lambda^+ & \kappa \leq \lambda \\ \kappa^+ & \mathrm{cf}(\kappa) \leq \lambda < \kappa \\ \kappa & \lambda < \mathrm{cf}(\kappa) \end{cases}$$

(For the third line use (1)(c)).

4. Shortly:

(1) Prove the *Hausdorff formula*:

$$(\kappa^+)^\lambda = \max\{\kappa^\lambda, \kappa^+\}.$$

Hint. This is easy if $\kappa^+ \leq \lambda$. So assume $\lambda < \kappa^+$, and put κ^+ instead of κ in (1)(b) of Question 1. (Recall that κ^+ is regular.)

(2) Rewrite the Hausdorff formula in the language of the aleph's $\aleph_\alpha$.

(3) Show that for each natural number n and each infinite λ, $(\aleph_n)^\lambda = \max\{2^\lambda, \aleph_n\}$.

The following fact (a proof can be found in Jech's book) is useful for Question 5: For each $\lambda \geq \mathrm{cf}(\kappa)$, $\kappa^\lambda = \left(\sup\{\mu^\lambda : \mu < \kappa\}\right)^{\mathrm{cf}(\kappa)}$.

5.

(1) Show that if $\aleph_\omega < 2^{\aleph_0}$, then $(\aleph_\omega)^{\aleph_0} = 2^{\aleph_0}$.
(2) Could it be that $\aleph_\omega = 2^{\aleph_0}$?
(3) Prove: $(\aleph_\omega)^{\aleph_1} = \max\{(\aleph_\omega)^{\aleph_0}, 2^{\aleph_1}\}$. (Use **4**(3).)
(4) Google "Shelah" and find his beautiful upper-bound on $(\aleph_\omega)^{\aleph_0}$.

Good luck!

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