A TASTE OF SET THEORY: EXERCISE NUMBER 5

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Recall that every natural number N has a unique representation in base 10, in the form

$$N = 10^{m_1}k_1 + 10^{m_2}k_2 + \cdots + 10^{m_n}k_n,$$

where $n \geq 1$, $m_1 > m_2 > \cdots > m_n$, and $k_1, \dots, k_n < 10$. Similarly, N has a unique representation in base q for any natural $q > 1$. Following is a beautiful transfinite analogue: Representation in base ω.

1. Using the results seen in class, prove the *Cantor Normal Form Theorem*:
Every ordinal $\alpha > 0$ has a unique representation in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_n$, and $k_1, \dots, k_n < \omega$ (i.e., are finite).

Hint: Prove the existence by induction on α, as follows:

(1) Check the case $\alpha = 1$.
(2) For $\alpha > 1$ let $\beta = \sup\{\gamma : \omega^\gamma \leq \alpha\}$. Then $\omega^\beta \leq \alpha$, too.
(3) There is $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$ for some δ. Show that δ is finite.

Prove the uniqueness of the representation by induction on α.

2. Define inductively $\alpha_0 = \omega$, and $\alpha_{n+1} = \omega^{\alpha_n}$ for $n \in \omega$, and $\epsilon_0 = \lim_{n\to\omega} \alpha_n$. Show that:

(1) $\omega^{\epsilon_0} = \epsilon_0$,
(2) ϵ_0 is the least ordinal α such that $\omega^\alpha = \alpha$.
(3) Find the Cantor normal form of ϵ_0 (see **Question 1**).

Zorn's Lemma is the following statement: Let A be a nonempty set, and $<$ be a partial ordering of A, such that each linearly ordered subset B of A is bounded in A.[1] Then there is a maximal element in A.[2]

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[1] I.e., there is $a \in A$ such that for each $b \in B$, $b \leq a$.

[2] I.e., there is $a \in A$ such that there is *no* $a' \in A$ satisfying $a < a'$.

It is not difficult to see that Zorn's Lemma implies AC (see booklet, Page 61).

3. Prove that AC implies Zorn's Lemma.

Hint: Assume that there is *no* maximal element in A. Since AC implies the Well-Ordering Principle, there is a well-ordering $\prec$ of A. Show that we can define $\mathbf{G} : \mathbf{ON} \to A$ such that:

(1) $\mathbf{G}(0)$ is the first element of A (with respect to $\prec$);
(2) For each successor ordinal $\alpha = \beta + 1$, $\mathbf{G}(\beta + 1)$ is the first element $a \in A$ (with respect to $\prec$) such that $\mathbf{G}(\beta) < a$; and
(3) For each limit ordinal α, $\mathbf{G}(\alpha)$ is an upper-bound (with respect to $<$) of $\{\mathbf{G}(\beta) : \beta < \alpha\}$.

Show that $\mathbf{G}$ is one-to-one.

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