A TASTE OF SET THEORY: EXERCISES 12 AND 13

BOAZ TSABAN

Undefined terminology and some hints can be found in Exercises 10 and 11 or in the lecture.

1. If there is a strongly unbounded $S \subseteq \mathbb{N}^\mathbb{N}$ of cardinality κ, then $\mathfrak{b} \le \kappa$ and $\mathrm{cf}(\kappa) \le \mathfrak{d}$.
Hint for the second assertion. If $D \subseteq \mathbb{N}^\mathbb{N}$ is dominating, then $S = \bigcup_{g \in D}\{f \in S : f \le^* g\}$.

Definition. A filter $\mathcal{F}$ on $\mathbb{N}$ is *free* if $\bigcap \mathcal{F} = \emptyset$.

2. Let $\mathcal{F}$ be a filter on $\mathbb{N}$.

(1) $\mathcal{F}$ is free if, and only if, $Fr \subseteq \mathcal{F}$.
(2) If $\mathcal{F}$ is free, then $\mathcal{F}$ is nonprincipal.
(3) If $\mathcal{F}$ is a nonprincipal *ultrafilter*, then $\mathcal{F}$ is free.

Definition. Let $\mathcal{F}$ be a filter on $\mathbb{N}$.
For $f, g \in \mathbb{N}^\mathbb{N}$, $f \le_\mathcal{F} g$ means: $\{n : f(n) \le g(n)\} \in \mathcal{F}$.
Similarly, $f <_\mathcal{F} g$ means $\{n : f(n) < g(n)\} \in \mathcal{F}$.
A set $B \subseteq \mathbb{N}^\mathbb{N}$ is *$\le_\mathcal{F}$-unbounded* if for each $f \in \mathbb{N}^\mathbb{N}$ there is $g \in B$ such that $g \not\le_\mathcal{F} f$.
$\mathfrak{b}(\mathcal{F})$ is the minimal cardinality of a $\le_\mathcal{F}$-unbounded set $B \subseteq \mathbb{N}^\mathbb{N}$.
$S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\} \subseteq \mathbb{N}^\mathbb{N}$ is a *$\mathfrak{b}(\mathcal{F})$-scale* if it is $\le_\mathcal{F}$-unbounded, and $\le_\mathcal{F}$-increasing with α (that is, for all $\alpha < \beta < \mathfrak{b}(\mathcal{F})$, $f_\alpha \le_\mathcal{F} f_\beta$).

3. Prove:

(1) $\le_{Fr} = \le^*$ (i.e., the two relations are the same), and $\mathfrak{b}(Fr) = \mathfrak{b}$.
(2) For each filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\le_\mathcal{F}$ is a reflexive and transitive relation on $\mathbb{N}^\mathbb{N}$.
  (b) There is a $\mathfrak{b}(\mathcal{F})$-scale.
  (c) $\mathfrak{b}(\mathcal{F})$ is regular.
(3) For each free filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\mathfrak{b} \le \mathfrak{b}(\mathcal{F})$.
  (b) Each $\mathfrak{b}(\mathcal{F})$-scale is strongly unbounded.
(4) Let $\mathcal{F}$ be the principal filter $\{A \subseteq \mathbb{N} : n \in A\}$:
  (a) What is $\mathfrak{b}(\mathcal{F})$?
  (b) Each $\mathfrak{b}(\mathcal{F})$-scale is bounded (with respect to $\le^*$).
(5) If $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $f \not\le_\mathcal{F} g \Leftrightarrow g <_\mathcal{F} f$.

Definition. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. $D \subseteq \mathbb{N}^\mathbb{N}$ is *$\le_\mathcal{F}$-dominating* if for each $g \in \mathbb{N}^\mathbb{N}$ there exists $f \in D$ such that $g \le_\mathcal{F} f$.
$\mathfrak{d}(\mathcal{F})$ is the minimal cardinality of a $\le_\mathcal{F}$-dominating subset of $\mathbb{N}^\mathbb{N}$.
$S = \{f_\alpha : \alpha < \mathfrak{d}(\mathcal{F})\} \subseteq \mathbb{N}^\mathbb{N}$ is a *$\mathfrak{d}(\mathcal{F})$-scale* if it is dominating, and for all $\alpha < \beta < \mathfrak{d}(\mathcal{F})$, $f_\beta \not\le_\mathcal{F} f_\alpha$.

4. Prove:

(1) $\mathfrak{d}(Fr) = \mathfrak{d}$.
(2) For each free filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\mathfrak{d}(\mathcal{F}) \leq \mathfrak{d}$.
  (b) There is a $\mathfrak{d}(\mathcal{F})$-scale.
  (c) Every $\mathfrak{d}(\mathcal{F})$-scale is strongly unbounded.
  (d) There is a strongly unbounded $S \subseteq \mathbb{N}^{\mathbb{N}}$ such that $|S| = \operatorname{cf}(\mathfrak{d}(\mathcal{F}))$.
(3) For each free filter $\mathcal{F}$ on $\mathbb{N}$, $\mathfrak{b} \leq \mathfrak{b}(\mathcal{F}) \leq \operatorname{cf}(\mathfrak{d}(\mathcal{F})) \leq \mathfrak{d}(\mathcal{F}) \leq \mathfrak{d}$.
(4) For each ultrafilter $\mathcal{F}$ on $\mathbb{N}$, $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}(\mathcal{F})$.

5. Assume that κ is minimal on which there is an entire measure. Prove:

(1) For each strongly unbounded $S \subseteq \mathbb{N}^{\mathbb{N}}$, $\kappa \neq |S|$.
Hint. Assume that μ is an entire measure on S.
For all $n, m \in \mathbb{N}$, define $U_{n,m} = \{f \in S : f(n) \leq m\}$.
For each n, $\bigcup_{m=1}^{\infty} U_{n,m} = S$, and the sets $U_{n,m}$ are increasing with m.
For each n there is $g(n) \in \mathbb{N}$ such that $\mu(U_{n,g(n)}) > 1 - \frac{1}{2^n}$. $\mu(\bigcup_{n=1}^{\infty} S \setminus U_{n,g(n)}) < 1$.
Let $V = \bigcap_{n=1}^{\infty} U_{n,g(n)} = \{f \in S : f \leq g\}$. $|V| < |S|$.
$\mu(S \setminus V) < 1$, thus $\mu(V) > 0$.
(2) For each free filter $\mathcal{F}$ on $\mathbb{N}$, $\kappa \notin \{\mathfrak{b}(\mathcal{F}), \operatorname{cf}(\mathfrak{d}(\mathcal{F})), \mathfrak{d}(\mathcal{F})\}$.
(3) $\kappa \notin \{\mathfrak{b}, \operatorname{cf}(\mathfrak{d}), \mathfrak{d}\}$.
(4) The *Banach-Kuratowski Theorem*: If there is an entire measure on $[0,1]$, then CH fails.

6 (Bonus). Prove:

(1) There is a set $D \subseteq \mathbb{N}^{\mathbb{N}}$ such that $|D| = \mathfrak{d}$ and for each $f \in \mathbb{N}^{\mathbb{N}}$, there is $g \in D$ such that $f(n) \leq g(n)$ for *all* n.
(2) Question 4(2)(a) is true for all filters.
(3) Question 4(2)(b) is true for all filters.
(4) Question 5(2) is true for all filters.

Hint for (4). Given $\mathcal{F}$, let $A = \bigcap \mathcal{F}$. Assume $A \neq \emptyset$. Consider the following cases separately: A is finite; A is infinite and $A \in \mathcal{F}$; A is infinite and $A \notin \mathcal{F}$. In the last case, define an appropriate filter on $\mathbb{N} \setminus A$, and consider the case that $\mathbb{N} \setminus A$ is finite and the case that it is infinite.

Good luck!

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: boaz.tsaban@weizmann.ac.il
URL: http://www.cs.biu.ac.il/~tsaban