A TASTE OF SET THEORY: EXERCISE 11

BOAZ TSABAN

Definition. For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means: $f(n) \leq g(n)$ for all but finitely many n (that is, there is m such that for all $n \geq m$, $f(n) \leq g(n)$). $B \subseteq \mathbb{N}^{\mathbb{N}}$ is *bounded* if there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in B$. $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$. Let $\mathfrak{c} = |\mathbb{R}|$.

1. Prove:

(1) $\aleph_1 \leq \mathfrak{b}$.
Hint. If $B = \{f_n : n \in \mathbb{N}\}$, consider $g(n) = \max\{f_0(n), f_1(n), \ldots, f_n(n)\}$.

(2) $\mathfrak{b} \leq \mathfrak{c}$.

Definition. $S = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-*scale* if it is unbounded, and $\leq^*$-increasing with α (that is, for all $\alpha < \beta < \mathfrak{b}$, $f_\alpha \leq^* f_\beta$).

$S \subseteq \mathbb{N}^{\mathbb{N}}$ is *strongly unbounded* if, for each $g \in \mathbb{N}^{\mathbb{N}}$, $|\{f \in S : f \leq^* g\}| < |S|$.

2. Prove:

(1) There is a $\mathfrak{b}$-scale.
Hint. Fix unbounded $B = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For each $\alpha < \mathfrak{b}$ take $g_\alpha \in \mathbb{N}^{\mathbb{N}}$ witnessing that $\{f_\beta, g_\beta : \beta < \alpha\}$ is bounded.

(2) Every $\mathfrak{b}$-scale is strongly unbounded.

(3) $\mathfrak{b}$ is regular.
Hint. Fix a $\mathfrak{b}$-scale $S = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For each cofinal $g : \beta \to \mathfrak{b}$, look at $\{f_{g(\alpha)} : \alpha < \beta\}$.

Definition. $D \subseteq \mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists $f \in D$ such that $g \leq^* f$. $\mathfrak{d}$ is the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$.

It is consistent that $\mathfrak{d}$ is singular.

Definition. $S = \{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{d}$-*scale* if it is dominating, and for all $\alpha < \beta < \mathfrak{d}$, $f_\beta \not\leq^* f_\alpha$.

3. Prove:

(1) There is a $\mathfrak{d}$-scale.
Hint. Similar to Question 3, but make sure that the resulting set is dominating.

(2) Every $\mathfrak{d}$-scale is strongly unbounded.

(3) There is a strongly unbounded $S \subseteq \mathbb{N}^{\mathbb{N}}$ such that $|S| = \mathrm{cf}(\mathfrak{d})$.
Hint. Fix a $\mathfrak{d}$-scale $S = \{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For a cofinal $g : \mathrm{cf}(\mathfrak{d}) \to \mathfrak{d}$, look at $\{f_{g(\alpha)} : \alpha < \mathrm{cf}(\mathfrak{d})\}$.

(4) $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.

Good luck!