A TASTE OF SET THEORY: EXERCISE 10

BOAZ TSABAN

Question 2 is not for credit. It is fine not to solve it. If you solve it, it will be checked but not graded.

Definition. $\mathcal{F}$ is a *filter* on S if:

(1) $\mathcal{F} \subseteq P(S)$;
(2) $S \in \mathcal{F}$;
(3) $\emptyset \notin \mathcal{F}$;
(4) $\forall A \in \mathcal{F} \forall B \subseteq S$, $A \subseteq B \longrightarrow B \in \mathcal{F}$;
(5) $\forall A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

Question 1. Prove (shortly!) that in the definition *filter*, each of the following changes can be made without leading to any mathematical difference:

(a) Item (2) can be replaced by: (2') $\mathcal{F} \neq \emptyset$.
(b) Item (3) can be replaced by: (3') $\mathcal{F} \neq P(S)$.
(c) Item (5) can be replaced by: (5') $\forall n \forall A_1, \ldots, A_n \in \mathcal{F}$, $A_1 \cap \cdots \cap A_n \in \mathcal{F}$.

Definition. A filter $\mathcal{F}$ on S is *maximal* if there is no filter $\mathcal{G}$ on S such that $\mathcal{F} \subsetneq \mathcal{G}$. A filter $\mathcal{F}$ on S is an *ultrafilter* if for each $A \subseteq S$, $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$.

Question 2. Let $\mathcal{F}$ be a filter on S. $\mathcal{F}$ is maximal if, and only if, $\mathcal{F}$ is an ultrafilter.

Definition. A filter $\mathcal{F}$ on S is *principal* if there is $A \subseteq S$ such that $\mathcal{F} = \{B \subseteq S : A \subseteq B\}$.

Question 3.

(1) If $\mathcal{F}$ is an ultrafilter on S and $A = B \cup C \in \mathcal{F}$, then $B \in \mathcal{F}$ or $C \in \mathcal{F}$.
(2) For each *principal ultrafilter* $\mathcal{F}$ on S, there is $x \in S$ such that $\mathcal{F} = \{A \subseteq S : x \in A\}$.

Definition. The *Fréchet filter* on an infinite set S is $Fr = \{A \subseteq S : |S \setminus A| < \aleph_0\}$.

Question 4. For each infinite S, the Fréchet filter on S is:

(1) A filter on S.
(2) A *nonprincipal* filter on S.
(3) Not an ultrafilter on S.
(4) There is an ultrafilter on S containing Fr.
Hint. Use transfinite induction or Zorn's Lemma to obtain a maximal filter $\mathcal{F}$ containing Fr, and use Question 2.
(5) Each filter on S which contains Fr is nonprincipal.

Definition. An *entire measure* μ on S is a function $\mu : P(S) \to [0, 1]$ such that:

(1) $\mu(\emptyset) = 0$, $\mu(S) = 1$.
(2) For each $A \subseteq B \subseteq S$, $\mu(A) \leq \mu(B)$.
(3) For each $s \in S$, $\mu(\{s\}) = 0$.
(4) For all pairwise disjoint sets $A_0, A_1, \ldots \subseteq S$, $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Say that μ is an *entire pseudomeasure* on S if it satisfies (1),(2),(3), and

(4') For each disjoint $A, B \subseteq S$, $\mu(A \cup B) = \mu A + \mu B$.

Question 5. For a *nonprincipal* ultrafilter $\mathcal{F}$ on an infinite set S, define $\mu_{\mathcal{F}} : P(S) \to \{0,1\}$ by $\mu_{\mathcal{F}}(A) = 1$ if $A \in \mathcal{F}$, and $\mu_{\mathcal{F}}(A) = 0$ otherwise.

(1) $\mu_{\mathcal{F}}$ is an entire pseudomeasure on S.
(2) For each infinite S, there is an entire pseudomeasure on S.
Hint. Question 4.

Good luck!

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: boaz.tsaban@weizmann.ac.il
URL: http://www.cs.biu.ac.il/~tsaban