A TASTE OF SET THEORY: EXERCISE 9

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Question 1. Prove:

(1) For each $\nu > \kappa$ and each μ, the property $\kappa \to (\kappa)^\nu_\mu$ is trivial.

(2) If $\sigma < \nu \le \mathrm{cf}(\lambda)$ and $\kappa \to (\lambda)^\nu_\mu$ holds, then $\kappa \to (\lambda)^\sigma_\mu$ holds as well.
[*Hint.* Given a coloring F of $[\kappa]^\sigma$, define a coloring G of $[\kappa]^\nu$ by $G(B) = F(\text{first } \sigma \text{ elements of } B)$. Assume that $M \in [\kappa]^\lambda$ is G-monochromatic. $\mathrm{otp}\, M \ge |M| = \lambda$, so take $N \subseteq M$ such that $\mathrm{otp}\, N = \lambda$. N is also G-monochromatic. If $A \in [N]^\sigma$, then A is bounded in N and can therefore be extended from above to a set $B \in [N]^\nu$.]

Question 2. Prove:

(1) For each infinite κ, λ, ν such that $\nu \le \lambda \le \kappa$ (and $\nu < \kappa$), and each $\mu > 1$, $\kappa \not\to (\lambda)^\nu_\mu$. (So ν must be finite for the property to make sense.)
[*Hint.* For $A, B \in [\kappa]^\nu$, define $A \sim B$ if $A \Delta B$ is finite. Fix a representative from each equivalence class, and define $F(A) = |A \Delta R| \bmod 2$ where R is the fixed representative of A's equivalence class.]

(2) What does (2) of Question 1 add to (1) of the current question?

Question 3. Prove:

(1) Each strictly increasing or strictly decreasing sequence in $\mathbb{R}$ (with respect to its usual order $<$) is countable.

(2) $\mathbb{R} \not\to (\aleph_1)^2_2$.
[*Hint.* Let $\prec$ be a wellordering of $\mathbb{R}$ and define $F : [\mathbb{R}]^2 \to 2$ by $F(\{x, y\}) = 1$ if $<$ and $\prec$ agree on x, y (i.e., "$x < y$ and $x \prec y$", or "$y < x$ and $y \prec x$"), and 0 if they disagree. If $M \in [\mathbb{R}]^{\aleph_1}$ is monochromatic for F, then $\alpha = \mathrm{otp}(M, \prec) \ge \aleph_1$. Use (1).]

(3) $2^{\aleph_0} \not\to (\aleph_1)^2_2$.

(4) $\aleph_1 \not\to (\aleph_1)^2_2$, so Ramsey's Theorem does not hold for $\aleph_1$ instead of $\aleph_0$.

Definitions. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *centered* if for each finite $\mathcal{F} \subseteq \mathcal{A}$, $\cap\mathcal{F}$ is infinite. For $A, B \subseteq \omega$, $A \subseteq^* B$ means that $A \setminus B$ is finite. $P \in [\omega]^\omega$ is a *pseudo-intersection* of $\mathcal{A}$ if for each $A \in \mathcal{A}$, $P \subseteq^* A$.

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Question 4. Prove:

(1) There is a countable centered family $\mathcal{A} \subseteq [\omega]^\omega$ such that $\cap\mathcal{A} = \emptyset$.
(2) Every countable centered family $\mathcal{A} \subseteq [\omega]^\omega$ has a pseudo-intersection.
[*Hint.* Let $\mathcal{A} = \{A_n : n \in \omega\}$, and for each n choose $a_n \in A_1 \cap A_2 \cdots \cap A_n$.]

Good luck!

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