A TASTE OF SET THEORY: PESSAH EXERCISE (NUMBER 7)

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In this exercise we will give some definitions and results which we did not get to in class. Thus, do not worry about its length. You are encouraged to look into Jech's book (Chapter 5) for more information.

Reminders. The letters $\alpha, \beta, \gamma, \delta$ always denote ordinals, whereas the letters $\kappa, \lambda, \mu, \sigma$ always denote cardinals.

For a cardinal κ, $\kappa^+ := \min\{\lambda : \kappa < \lambda\}$, so that for each α, $\aleph_{\alpha+1} = (\aleph_\alpha)^+$.

The *Generalized Continuum Hypothesis (GCH)* is the assertion:

$$(\forall \kappa)\ 2^\kappa = \kappa^+.$$

(equivalently, $(\forall \alpha)\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.)

Question 0 (Curiose). The *beth function* is defined by transfinite recursion:

$$\begin{aligned} \beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\gamma &= \sup\{\beth_\alpha : \alpha < \gamma\} \quad \text{for a limit } \gamma. \end{aligned}$$

Rewrite GCH using the beth function.

Question 1. Supply very short proofs to each of the following.

(1) Assume that $\lambda < \mathrm{cf}(\kappa)$. Show:
  - (a) ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$.
  - (b) $\kappa^\lambda \le \kappa \cdot \sup\{\mu^\lambda : \mu < \kappa\}$ (Use (a)).
  - (c) $\kappa^\lambda \le \kappa \cdot \sup\{2^\mu : \mu < \kappa\}$ (Use (b)).

(2) Assume GCH. Show that for all infinite cardinals κ and λ:

$$\kappa^\lambda = \begin{cases} \lambda^+ & \kappa \le \lambda \\ \kappa^+ & \mathrm{cf}(\kappa) \le \lambda < \kappa \\ \kappa & \lambda < \mathrm{cf}(\kappa) \end{cases}$$

(For the third line use (1)(c)).

Question 2. Shortly:

(1) Prove the *Hausdorff formula*:

$$(\kappa^+)^\lambda = \max\{\kappa^\lambda, \kappa^+\}.$$

[*Hint.* This is easy if $\kappa^+ \le \lambda$. So assume $\lambda < \kappa^+$, and put κ^+ instead of κ in (1)(b) of Question 1. (Recall that κ^+ is regular.)]

(2) Rewrite the Hausdorff formula in the language of the aleph's $\aleph_\alpha$.

(3) Show that for each natural number n and each infinite λ, $(\aleph_n)^\lambda = \max\{2^\lambda, \aleph_n\}$.

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The following lemma will be useful in some of the coming exercises.

Lemma. For each limit cardinal κ:

(1) $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.
(2) For each $\lambda \geq \mathrm{cf}(\kappa)$, $\kappa^\lambda = \big(\sup\{\mu^\lambda : \mu < \kappa\}\big)^{\mathrm{cf}(\kappa)}$.

Proof. Write $\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha$ where each $\kappa_\alpha < \kappa$.

Proof of (1): Clearly,

$$(2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^{\kappa\cdot\mathrm{cf}(\kappa)} = 2^\kappa.$$

On the other hand,

$$2^\kappa = 2^{\sum_\alpha \kappa_\alpha} = \prod_{\alpha<\mathrm{cf}(\kappa)} 2^{\kappa_\alpha} \leq \prod_{\alpha<\mathrm{cf}(\kappa)} 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)}.$$

Proof of (2): Let $\sigma = \sup\{\mu^\lambda : \mu < \kappa\}$. Clearly, $\sigma \leq \kappa^\lambda$ and therefore

$$\sigma^{\mathrm{cf}(\kappa)} \leq (\kappa^\lambda)^{\mathrm{cf}(\kappa)} = \kappa^{\lambda\cdot\mathrm{cf}(\kappa)} = \kappa^\lambda.$$

On the other hand,

$$\kappa^\lambda = (\sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha)^\lambda \leq (\prod_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha)^\lambda = \prod_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha^\lambda \leq \prod_{\alpha<\mathrm{cf}(\kappa)} \sigma = \sigma^{\mathrm{cf}(\kappa)}. \quad \square$$

Question 3. Solve the following by quoting general facts.

(1) Prove: $\prod_{n\in\mathbb{N}} \aleph_n = (\aleph_\omega)^{\aleph_0}$.
(2) Show that if $\aleph_\omega < 2^{\aleph_0}$, then $(\aleph_\omega)^{\aleph_0} = 2^{\aleph_0}$.
(3) Could it be that $\aleph_\omega = 2^{\aleph_0}$?
(4) Prove: $(\aleph_\omega)^{\aleph_1} = \max\{(\aleph_\omega)^{\aleph_0}, 2^{\aleph_1}\}$. (Notice Question 2(3).)
(5) Google "Shelah" and find his beautiful upper-bound on $(\aleph_\omega)^{\aleph_0}$.

Question 4. Assume that GCH holds. Prove:

(1) $2^{<\kappa} = \kappa$ for all κ.
(2) $\kappa^{<\kappa} = \kappa$ for all regular κ.
(3) Can (2) hold for singular κ? Why?

Good luck!

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