A TASTE OF SET THEORY: EXERCISE NUMBER 5

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1. Prove that every infinite cardinal number is a limit ordinal.

[In other words: If $\kappa \geq \omega$ is a cardinal and $\alpha < \kappa$ is an ordinal, then $\alpha + 1 < \kappa$ (addition here is ordinal addition). You can use without proof any fact concerning ordinals (but not cardinals) from the booklet *Cantor's Dream* to show that. The answer must be short.]

2. Read the proof of the Product Theorem available in the course's homepage. Did you understand it? Do you find it complete, or would you like to see more details proved?

3. Let $\alpha > 0$ be a limit ordinal. Prove:

(1) If $A \subseteq \alpha$ and $\sup A = \alpha$, then $\operatorname{otp}(A) \geq \operatorname{cf}(\alpha)$.

[$\operatorname{otp}(A)$ denotes the order type of A with respect to the usual $<$ on ordinals.]

(2) If $(\beta_\xi : \xi < \gamma)$ is a nondecreasing γ-sequence of ordinals smaller than α such that $\lim_{\xi\to\gamma} \beta_\xi = \alpha$, then:
   (a) γ is a limit ordinal, and
   (b) $\operatorname{cf}(\gamma) = \operatorname{cf}(\alpha)$.

4. Let α be a limit ordinal. Say that a function $f : \beta \to \alpha$ is *unbounded* in α if $\sup f[\beta] = \alpha$.

Prove:

(1) $\operatorname{cf}(\alpha) = \min\{\beta : \exists \text{ increasing unbounded } f : \beta \to \alpha\}$. (*Trivial.*)

(2) For each unbounded $f : \beta \to \alpha$, there is $\gamma \leq \beta$ and an *increasing* unbounded $g : \gamma \to \alpha$ such that $\operatorname{im}(g) \subseteq \operatorname{im}(f)$.

[*Hint.* By induction on $\xi < \beta$, define $g(\xi) = \min(f[\beta] \setminus \sup\{g(\eta)+1 : \eta < \xi\})$ as long as this is possible.]

(3) Use (1) and (2) to show that: $\operatorname{cf}(\alpha) = \min\{\beta : \exists \text{ unbounded } f : \beta \to \alpha\}$.

5. Show that for each limit ordinal α, $\beta = \operatorname{cf}(\alpha)$ is a cardinal.

[*Hint.* If β is not a cardinal, then $|\beta| < \beta$. By Question 4, $\operatorname{cf}(\beta) \leq |\beta| < \beta$, which is a contradiction if $\beta = \operatorname{cf}(\alpha)$.]

Good luck!

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