A TASTE OF SET THEORY: EXERCISE NUMBER 3

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1. Prove that for each ordinal α there is a limit ordinal $\delta > \alpha$.
[*Hint:* Show that $\alpha + \omega$ is a limit ordinal.]

2. Find α, β, γ such that $\alpha < \beta$ but still:

(1) $\alpha + \gamma = \beta + \gamma$.
(2) $\alpha \cdot \gamma = \beta \cdot \gamma$. (Not necessarily the same as (1).)

Why does this not contradict what we have studied in class?

Recall that every natural number N has a unique representation in base 10, in the form

$$N = 10^{m_1}k_1 + 10^{m_2}k_2 + \cdots + 10^{m_n}k_n,$$

where $n \geq 1$, $m_1 > m_2 > \cdots > m_n$, and $k_1, \ldots, k_n < 10$. Similarly, N has a unique representation in base q for any natural $q > 1$. Following is a beautiful transfinite analogue: Representation in base ω!

3. Using the results seen in class, prove the *Cantor Normal Form Theorem*: Every ordinal $\alpha > 0$ has a unique representation in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_n$, and $k_1, \ldots, k_n < \omega$ (i.e., are finite).

Hint:

(1) Prove the existence by induction on α, as follows:
    (a) Check the case $\alpha = 1$.
    (b) For $\alpha > 1$ let $\beta = \sup\{\gamma : \omega^\gamma \leq \alpha\}$. Then $\omega^\beta \leq \alpha$, too.
    (c) Use results from class to show that there is $\rho < \omega^\beta$ such that $\omega^\beta \cdot \delta + \rho$ for some δ. Show that δ is finite.
(2) Prove the uniqueness of the representation by induction on α.

4 (Bonus). Define inductively $\alpha_0 = \omega$, and $\alpha_{n+1} = \omega^{\alpha_n}$ for $n \in \omega$, and $\epsilon_0 = \lim_{n\to\omega} \alpha_n$. Show that:

(1) $\omega^{\epsilon_0} = \epsilon_0$,
(2) ϵ_0 is the least ordinal α such that $\omega^\alpha = \alpha$.
(3) Find the Cantor normal form of ϵ_0 (see **Question 3**).

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