A TASTE OF SET THEORY: EXERCISE NUMBER 2

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1. Prove the following facts:

(1) If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.
(2) If α, β are distinct ordinals and $\alpha \subseteq \beta$, then $\alpha \in \beta$.
[*Hint*: Let γ be the least element in $\beta \setminus \alpha$. Show that $\alpha = \gamma$.]
(3) If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
[*Hint*: Show that $\gamma := \alpha \cap \beta$ is an ordinal. Explain why $\gamma = \alpha$ or $\gamma = \beta$.]

2. Using the previous exercise and the definitions, prove that for each nonempty class C of ordinals:

(1) $\bigcap C$ is an ordinal.
(2) $\bigcap C \in C$.
(3) $\bigcap C = \inf C$.

3. Prove the following facts:

(1) For each ordinal α, $\alpha + 1$ is an ordinal.
(2) α is a limit ordinal (i.e., not a successor ordinal) if, and only if, $\alpha = \sup\{\beta : \beta < \alpha\} = \sup \alpha$.

4. Recall that a set X is *inductive* if: $\emptyset \in X$, and for each $x \in X$, $x \cup \{x\} \in X$ too. Define

$$\omega = \bigcap \{X : X \text{ is inductive}\}.$$

Prove the following facts:

(1) If a set X is inductive, then $X \cap \mathit{Ord}$ is inductive.
(2) ω is an ordinal.
(3) ω is a *limit* ordinal.
(4) ω is the *least* nonzero limit ordinal.

Good luck!

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Date: 7 Mar 05 CE.