A TASTE OF SET THEORY: EXERCISE 9

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Read the whole exercise, but solve only questions 1,2,5 (you can quote the unsolved questions in your solutions).

Undefined terminology is as in the new link at the webpage, or in the previous exercise.

1. If there is a strongly unbounded $S \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality κ, then $\mathfrak{b} \le \kappa$ and $\operatorname{cf}(\kappa) \le \mathfrak{d}$.
Hint for the second assertion. If $D \subseteq \mathbb{N}^{\mathbb{N}}$ is dominating, then $S = \bigcup_{g \in D} \{f \in S : f \le^* g\}$.

Definitions. A filter $\mathcal{F}$ on $\mathbb{N}$ is *free* if $\bigcap \mathcal{F} = \emptyset$. *Fr* is the Fréchet filter on $\mathbb{N}$.

2. Let $\mathcal{F}$ be a filter on $\mathbb{N}$.

(1) $\mathcal{F}$ is free if, and only if, $Fr \subseteq \mathcal{F}$.
(2) If $\mathcal{F}$ is free, then $\mathcal{F}$ is nonprincipal.
(3) If $\mathcal{F}$ is a nonprincipal *ultrafilter*, then $\mathcal{F}$ is free.

Definitions. Let $\mathcal{F}$ be a filter on $\mathbb{N}$.
For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \le_{\mathcal{F}} g$ means: $\{n : f(n) \le g(n)\} \in \mathcal{F}$.
Similarly, $f <_{\mathcal{F}} g$ means $\{n : f(n) < g(n)\} \in \mathcal{F}$.
A set $B \subseteq \mathbb{N}^{\mathbb{N}}$ is *$\le_{\mathcal{F}}$-unbounded* if for each $f \in \mathbb{N}^{\mathbb{N}}$ there is $g \in B$ such that $g \not\le_{\mathcal{F}} f$.
$\mathfrak{b}(\mathcal{F})$ is the minimal cardinality of a $\le_{\mathcal{F}}$-unbounded set $B \subseteq \mathbb{N}^{\mathbb{N}}$.
$S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *$\mathfrak{b}(\mathcal{F})$-scale* if it is $\le_{\mathcal{F}}$-unbounded, and $\le_{\mathcal{F}}$-increasing with α (that is, for all $\alpha < \beta < \mathfrak{b}(\mathcal{F})$, $f_\alpha \le_{\mathcal{F}} f_\beta$).

3. Prove:

(1) $\le_{Fr} = \le^*$ (i.e., the two relations are the same), and $\mathfrak{b}(Fr) = \mathfrak{b}$.
(2) For each filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\le_{\mathcal{F}}$ is a reflexive and transitive relation on $\mathbb{N}^{\mathbb{N}}$.
  (b) There is a $\mathfrak{b}(\mathcal{F})$-scale.
  (c) $\mathfrak{b}(\mathcal{F})$ is regular.
(3) For each free filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\mathfrak{b} \le \mathfrak{b}(\mathcal{F})$.
  (b) Each $\mathfrak{b}(\mathcal{F})$-scale is strongly unbounded.
(4) Let $\mathcal{F}$ be the principal filter $\{A \subseteq \mathbb{N} : n \in A\}$:
  (a) What is $\mathfrak{b}(\mathcal{F})$?
  (b) Each $\mathfrak{b}(\mathcal{F})$-scale is bounded (with respect to $\le^*$).
(5) If $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $f \not\le_{\mathcal{F}} g \Leftrightarrow g <_{\mathcal{F}} f$.

Definition. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. $D \subseteq \mathbb{N}^{\mathbb{N}}$ is *$\le_{\mathcal{F}}$-dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists $f \in D$ such that $g \le_{\mathcal{F}} f$.
$\mathfrak{d}(\mathcal{F})$ is the minimal cardinality of a $\le_{\mathcal{F}}$-dominating subset of $\mathbb{N}^{\mathbb{N}}$.
$S = \{f_\alpha : \alpha < \mathfrak{d}(\mathcal{F})\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *$\mathfrak{d}(\mathcal{F})$-scale* if it is dominating, and for all $\alpha < \beta < \mathfrak{d}(\mathcal{F})$, $f_\beta \not\le_{\mathcal{F}} f_\alpha$.

4. Prove:

(1) $\mathfrak{d}(Fr) = \mathfrak{d}$.

(2) For each free filter $\mathcal{F}$ on $\mathbb{N}$:
  (a) $\mathfrak{d}(\mathcal{F}) \leq \mathfrak{d}$.
  (b) There is a $\mathfrak{d}(\mathcal{F})$-scale.
  (c) Every $\mathfrak{d}(\mathcal{F})$-scale is strongly unbounded.
  (d) There is a strongly unbounded $S \subseteq \mathbb{N}^\mathbb{N}$ such that $|S| = \mathrm{cf}(\mathfrak{d}(\mathcal{F}))$.
(3) For each free filter $\mathcal{F}$ on $\mathbb{N}$, $\mathfrak{b} \leq \mathfrak{b}(\mathcal{F}) \leq \mathrm{cf}(\mathfrak{d}(\mathcal{F})) \leq \mathfrak{d}(\mathcal{F}) \leq \mathfrak{d}$.
(4) For each ultrafilter $\mathcal{F}$ on $\mathbb{N}$, $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}(\mathcal{F})$.

5. Assume that κ is minimal on which there is an entire measure. Prove:
(1) For each strongly unbounded $S \subseteq \mathbb{N}^\mathbb{N}$, $\kappa \neq |S|$.
Hint. Assume that μ is an entire measure on S.
For all $n, m \in \mathbb{N}$, define $U_{n,m} = \{f \in S : f(n) \leq m\}$.
For each n, $\bigcup_{m=1}^\infty U_{n,m} = S$, and the sets $U_{n,m}$ are increasing with m.
For each n there is $g(n) \in \mathbb{N}$ such that $\mu(U_{n,g(n)}) > 1 - \frac{1}{2^n}$. $\mu(\bigcup_{n=1}^\infty S \setminus U_{n,g(n)}) < 1$.
Let $V = \bigcap_{n=1}^\infty U_{n,g(n)} = \{f \in S : f \leq g\}$. $|V| < |S|$.
$\mu(S \setminus V) < 1$, thus $\mu(V) > 0$.
(2) For each free filter $\mathcal{F}$ on $\mathbb{N}$, $\kappa \notin \{\mathfrak{b}(\mathcal{F}), \mathrm{cf}(\mathfrak{d}(\mathcal{F})), \mathfrak{d}(\mathcal{F})\}$.
(3) $\kappa \notin \{\mathfrak{b}, \mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$.
(4) The *Banach-Kuratowski Theorem*: If there is an entire measure on $[0,1]$, then CH fails.

6. Prove:
(1) There is a set $D \subseteq \mathbb{N}^\mathbb{N}$ such that $|D| = \mathfrak{d}$ and for each $f \in \mathbb{N}^\mathbb{N}$, there is $g \in D$ such that $f(n) \leq g(n)$ for *all* n.
(2) Question 4(2)(a) is true for all filters.
(3) Question 4(2)(b) is true for all filters.
(4) Question 5(2) is true for all filters.

Hint for (4). Given $\mathcal{F}$, let $A = \bigcap \mathcal{F}$. Assume $A \neq \emptyset$. Consider the following cases separately: A is finite; A is infinite and $A \in \mathcal{F}$; A is infinite and $A \notin \mathcal{F}$. In the last case, define an appropriate filter on $\mathbb{N} \setminus A$, and consider the case that $\mathbb{N} \setminus A$ is finite and the case that it is infinite.

Good luck!

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