

A TASTE OF SET THEORY: EXERCISE 8

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Definition. For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means: $f(n) \leq g(n)$ for all but finitely many n (that is, there is m such that for all $n \geq m$, $f(n) \leq g(n)$). $B \subseteq \mathbb{N}^{\mathbb{N}}$ is *bounded* if there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in B$. \mathfrak{b} is the minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$. Let $\mathfrak{c} = |\mathbb{R}|$.

1. Prove:

$$(1) \aleph_1 \leq \mathfrak{b}.$$

Hint. If $B = \{f_n : n \in \mathbb{N}\}$, consider $g(n) = \max\{f_0(n), f_1(n), \dots, f_n(n)\}$.

$$(2) \mathfrak{b} \leq \mathfrak{c}.$$

Definition. $S = \{f_{\alpha} : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *\mathfrak{b} -scale* if it is unbounded, and \leq^* -increasing with α (that is, for all $\alpha < \beta < \mathfrak{b}$, $f_{\alpha} \leq^* f_{\beta}$).

$S \subseteq \mathbb{N}^{\mathbb{N}}$ is *strongly unbounded* if, for each $g \in \mathbb{N}^{\mathbb{N}}$, $|\{f \in S : f \leq^* g\}| < |S|$.

2. Prove:

$$(1) \text{ There is a } \mathfrak{b}\text{-scale.}$$

Hint. Fix unbounded $B = \{f_{\alpha} : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For each $\alpha < \mathfrak{b}$ take $g_{\alpha} \in \mathbb{N}^{\mathbb{N}}$ witnessing that $\{f_{\beta}, g_{\beta} : \beta < \alpha\}$ is bounded.

$$(2) \text{ Every } \mathfrak{b}\text{-scale is strongly unbounded.}$$

$$(3) \mathfrak{b} \text{ is regular.}$$

Hint. Fix a \mathfrak{b} -scale $S = \{f_{\alpha} : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For each cofinal $g : \beta \rightarrow \mathfrak{b}$, look at $\{f_{g(\alpha)} : \alpha < \beta\}$.

Definition. $D \subseteq \mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists $f \in D$ such that $g \leq^* f$. \mathfrak{d} is the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$.

It is consistent that \mathfrak{d} is singular.

Definition. $S = \{f_{\alpha} : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *\mathfrak{d} -scale* if it is dominating, and for all $\alpha < \beta < \mathfrak{d}$, $f_{\beta} \not\leq^* f_{\alpha}$.

3. Prove:

$$(1) \text{ There is a } \mathfrak{d}\text{-scale.}$$

Hint. Similar to Question 3, but make sure that the resulting set is dominating.

$$(2) \text{ Every } \mathfrak{d}\text{-scale is strongly unbounded.}$$

$$(3) \text{ There is a strongly unbounded } S \subseteq \mathbb{N}^{\mathbb{N}} \text{ such that } |S| = \text{cf}(\mathfrak{d}).$$

Hint. Fix a \mathfrak{d} -scale $S = \{f_{\alpha} : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$. For a cofinal $g : \text{cf}(\mathfrak{d}) \rightarrow \mathfrak{d}$, look at $\{f_{g(\alpha)} : \alpha < \text{cf}(\mathfrak{d})\}$.

$$(4) \aleph_1 \leq \text{cf}(\mathfrak{b}) = \mathfrak{b} \leq \text{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}.$$

Good luck!

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