A TASTE OF SET THEORY: EXERCISE 6

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1. Let $\alpha > 0$ be a limit ordinal, and $A \subseteq \alpha$ be such that $\sup A = \alpha$. Prove that the order type of A is $\geq \operatorname{cf}(\alpha)$.

2. Assume that $\lambda < \operatorname{cf}(\kappa)$ and for each $\alpha < \lambda$, $|X_\alpha| < \kappa$. Show that $|\bigcup_{\alpha<\lambda} X_\alpha| < \kappa$ (in particular, $\bigcup_{\alpha<\lambda} X_\alpha \neq \kappa$).

Definitions. A *coloring* of (the elements of) a set A with λ many colors (λ may be finite or infinite) is a function $f : A \to \lambda$ (so that each element $a \in A$ is given the "color" $f(a)$). A subset $C \subseteq A$ is *monochromatic* if f is constant on C (i.e., all elements in C were given the same color).

Consider the following *Ramsey-theoretic* assertion concerning A, λ, and another cardinal κ, which may be true or false:

$A \to (\kappa)^1_\lambda$: For each coloring f of A with λ colors, there is a monochromatic $C \subseteq A$ such that $|C| = \kappa$.

3. Prove: A cardinal κ is regular if, and only if, $\kappa \to (\kappa)^1_\lambda$ for all $\lambda < \kappa$.

4. Prove:
(1) For each limit ordinal α, $\operatorname{cf}(\aleph_\alpha) = \operatorname{cf}(\alpha)$.
(2) Compute: $\operatorname{cf}(\aleph_{\aleph_1})$, $\operatorname{cf}(\aleph_{\aleph_{\aleph_1}})$.

Good luck!

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