A TASTE OF SET THEORY: EXERCISE 5

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1. Prove Lemma 10.19 on page 30 of Kunen's book.

Definitions. An *n-ary* function on a set A is an $f : A^n \to A$ if $n > 0$, and a constant if $n = 0$. A set $B \subseteq A$ is *closed under* f if the set $f[B^n] = \{f(v) : v \in B^n\}$ is a subset of B (or $f \in B$ if $n = 0$). f is a *finitary* function if it is n-ary for some $n \in \mathbb{N}$. For a family F of finitary functions on A and $B \subseteq A$, the *closure of* B under F is the smallest (with respect to inclusion) $C \subseteq A$ such that $B \subseteq C$ and C is closed under all functions from F. (Note that $C = \bigcap\{D : B \subseteq D \subseteq A \text{ and } D \text{ is closed under } F\}$.)

2. Assume that $B \subseteq A$, $|B| \leq \kappa$, κ is infinite, and F is a family of at most κ many finitary functions on A. Then the closure of B under F has cardinality $\leq \kappa$.
Hint. For $D \subseteq A$, and an n-ary $f \in F$, define $f * D = f[D^n]$ if $n > 0$, and $f * D = \{f\}$ if $n = 0$. Prove: If $|D| \leq \kappa$, then $|f * D| \leq \kappa$. Set $C_0 = D$, $C_{n+1} = C_n \cup \bigcup\{f * C_n : f \in F\}$, and $C_\omega = \bigcup_{n\in\mathbb{N}} C_n$. C_ω is the closure of B under F.

3. Prove:
(1) Let G be a group and D be any infinite subset of G. Then the subgroup of G generated by D (that is, the smallest subgroup of G containing D) has the same cardinality as D.
(2) What can you say about (1) when D is finite? (Give an example showing that your answer is optimal.)
(3) Let $\aleph_\alpha = |G|$ and $\kappa = |\alpha|$. There are at least κ many nonisomorphic subgroups of G.
(4) If CH fails, then there is a subgroup of the additive group $\mathbb{Z}^\mathbb{N}$, which is not isomorphic to $\mathbb{Z}^\mathbb{N}$ and not to $\mathbb{Z}^n$ for any n.

Definition. Recall that for sets A, B, ${}^AB$ is the set of all functions $f : A \to B$. For cardinals κ, λ, define *cardinal exponentiation* by

$$\kappa^\lambda = |{}^\lambda\kappa|.$$

4. Prove, by defining a bijection between the relevant sets, that the cardinal arithmetic operations satisfy:

$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$$

for all cardinals κ, λ, μ.

Good luck!

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