A TASTE OF SET THEORY: EXERCISE 4

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1. Prove that AC implies the Well Ordering Principle.

Hint: Let A be given. Let f be a choice function on $\mathcal{F} = P(A) \setminus \{\emptyset\}$. Fix $b \notin A$.

Use f and the Recursion Theorem to define $G : ON \to V$ such that $G(\alpha) \in A \setminus \{G(\beta) : \beta < \alpha\}$ if this set is nonempty, and $G(\alpha) = b$ otherwise. Show that G induces a bijection g from $\gamma = \sup\{\alpha : G(\alpha) \neq b\}$ (why is this a set?) to A. Order A by $a_1 < a_2$ iff $g^{-1}(a_1) < g^{-1}(a_2)$.

Look up *Partially ordered set* in Wikipedia. *Zorn's Lemma* is the following statement: Let A be a nonempty set partially ordered by $\leq$, such that each linearly ordered subset B of A is bounded in A.[1] Then there is a maximal element in A.[2]

It is not difficult to see that Zorn's Lemma implies AC (see booklet, Page 61).

2. Prove that AC implies Zorn's Lemma.

Hint: Assume that there is *no* maximal element in A. Since AC implies the Well-Ordering Principle, there is a well-ordering $\prec$ of A. Show that we can define $\mathbf{G} : \mathbf{ON} \to A$ such that:

(1) $\mathbf{G}(0)$ is the first element of A (with respect to $\prec$);
(2) For each successor ordinal $\alpha = \beta + 1$, $\mathbf{G}(\beta + 1)$ is the first element $a \in A$ (with respect to $\prec$) such that $\mathbf{G}(\beta) < a$; and
(3) For each limit ordinal α, $\mathbf{G}(\alpha)$ is an upper-bound (with respect to $<$) of $\{\mathbf{G}(\beta) : \beta < \alpha\}$.

Show that $\mathbf{G}$ is one-to-one.

3. Prove the following assertions:

(a) Assume that $f : \alpha \to ON$ is an increasing function (that is, for all $\beta < \gamma < \alpha$, $f(\beta) < f(\gamma)$). Then for each $\beta < \alpha$, $f(\beta) \geq \beta$.

[1] I.e., there is $a \in A$ such that for each $b \in B$, $b \leq a$.

[2] I.e., there is $a \in A$ such that there is *no* $a' \in A$ satisfying $a < a'$.

(b) Let α be an ordinal and $F \subseteq \alpha$. Then the order type of F is $\leq \alpha$.

Hint for (b): Let β be the order type of F, and $f : \beta \to F$ be an order isomorphism. Then f is increasing.

4.Prove:

(1) $A \preceq B \Leftrightarrow |A| \leq |B|$. (*Hint for* $\Rightarrow$: Question 3.)
(2) $A \prec B \Leftrightarrow |A| < |B|$.
(3) $A \approx B \Leftrightarrow |A| = |B|$.

Good luck!

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