A TASTE OF SET THEORY: EXERCISE 3

BOAZ TSABAN

1. Find α, β, γ such that $\alpha < \beta$ but still $\alpha \cdot \gamma = \beta \cdot \gamma$. (Supply detailed calculations.)

Recall the previous exercise sheet, in which we defined subtraction of ordinals. The next exercise defines *division with remainder* of ordinals. In this exercise, we can denote the γ by α/β, and the δ by "$\alpha \bmod \beta$".

2. Prove: For each α and each $\beta > 0$, there are unique γ, δ such that $\delta < \beta$, and $\alpha = \beta\gamma + \delta$.

Recall that every natural number N has a unique representation in base 10, in the form

$$N = 10^{m_1}k_1 + 10^{m_2}k_2 + \cdots + 10^{m_n}k_n,$$

where $n \geq 1$, $m_1 > m_2 > \cdots > m_n$, and $0 < k_1, \ldots, k_n < 10$. Similarly, N has a unique representation in base q for any natural $q > 1$. Following is a beautiful transfinite analogue: Representation in base ω.

3. Prove the *Cantor Normal Form Theorem*:
Every ordinal $\alpha > 0$ has a unique representation in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_n$, and $0 < k_1, \ldots, k_n < \omega$.

Hint: Prove the existence by induction on α, as follows:

(1) Check the case $\alpha = 1$.
(2) For $\alpha > 1$ let $\beta = \sup\{\gamma : \omega^\gamma \leq \alpha\}$. Then $\omega^\beta \leq \alpha$, too.
(3) There is $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$ for some δ. Show that δ is finite.

Prove the uniqueness of the representation by induction on α.

4. Define inductively $\alpha_0 = \omega$, and $\alpha_{n+1} = \omega^{\alpha_n}$ for $n \in \omega$, and $\epsilon_0 = \lim_{n\to\omega} \alpha_n$. Show that:

(1) $\omega^{\epsilon_0} = \epsilon_0$,
(2) ϵ_0 is the least ordinal α such that $\omega^\alpha = \alpha$.
(3) Find the Cantor normal form of ϵ_0.

Good luck!

Department Mathematics, Bar-Ilan University; and Department Mathematics, Weizmann Institute of Science

E-mail address: `tsaban@math.biu.ac.il`

URL: `http://www.cs.biu.ac.il/~tsaban`