# A TASTE OF SET THEORY: EXERCISE 2

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**1.** Assume that $\alpha, \beta$ are ordinals, and $\beta \subsetneq \alpha$. Prove that $\beta$ is the first element in $\alpha \setminus \beta$.

*Definition.* A set $A \subseteq \alpha$ is *bounded* in $\alpha$ if there is $\beta \in \alpha$ such that for each $\gamma \in A$, we have that $\gamma < \beta$. If $A \subseteq \alpha$ is not bounded in $\alpha$, we say that it is *unbounded* (in $\alpha$).

**2.** Prove that the following properties are equivalent for an ordinal $\alpha > 0$:

(1) $\alpha$ is a limit ordinal.
(2) $\alpha = \sup\{\xi : \xi < \alpha\}$.
(3) For each unbounded $A \subseteq \alpha$, $\alpha = \sup A$.

**3.** Let $\alpha, \beta, \gamma$ be ordinals. Prove:

(1) $\alpha + 1 = S(\alpha)$.
(2) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

**4.** Prove that for each ordinal $\alpha$ there is a limit ordinal $\delta > \alpha$.
[*Hint:* Show that $\alpha + \omega$ is a limit ordinal.]

In the following question, we can denote the $\delta$ by $\beta - \alpha$, and thus we have defined *subtraction of ordinals*.

**5.** Prove: If $\alpha < \beta$, then there is a unique ordinal $\delta$ such that $\alpha + \delta = \beta$.

*Good luck!*

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