A TASTE OF SET THEORY: FINAL EXERCISE

BOAZ TSABAN

It's more fun solving without hints. When needed, hints are available on the next page.

1. Prove the *Mostowski Collapse Theorem*:
Assume that E is well-founded, set-like, and extensional on the class P. Then:

(1) There is a transitive class M such that $(M, \in) \cong (P, E)$.
(2) M and the isomorphism are unique.

Terminology. The transitive class M in the Mostowski Collapse Theorem will be called the *collapse* of (P, E).

2. Assume that E well-orders P, and is set-like and extensional on P. Prove:

(1) If P is a set, then the collapse of (P, E) is an ordinal, namely, the order type of (P, E).
(2) If P is a proper class, then the collapse of (P, E) is ON, the class of all ordinal numbers.

3. Let κ be a measurable cardinal, and $\mathcal{U}$ be a nonprincipal κ-complete ultrafilter on κ. Prove:

(1) For each $f \in V^\kappa$, $\{g \in V^\kappa : g =^* f\}$ is a proper class.
(2) E is well-defined, set-like, and well-founded on $V^\kappa/\mathcal{U}$.

4. Prove **Łos' Theorem**: Let κ be a measurable cardinal, and $\mathcal{U}$ be a measure on κ. For each formula $\varphi(x_1, \dots, x_n)$, and all $f_1, \dots, f_n \in V^\kappa$,

$$V^\kappa/\mathcal{U} \models \varphi([f_1], \dots, [f_n]) \quad \Leftrightarrow \quad \{\alpha < \kappa : \varphi(f_1(\alpha), \dots, f_n(\alpha))\} \in \mathcal{U}.$$

Definition. $i : V \to M$ is an *elementary embedding* if it is injective, and for each formula φ and all $a_1, \dots, a_n \in V$, $V \models \varphi(a_1, \dots, a_n) \Leftrightarrow i[V] \models \varphi(i(a_1), \dots, i(a_n)) \Leftrightarrow M \models \varphi(i(a_1), \dots, i(a_n))$.

We have proved that for each measurable κ, there is a transitive M and an elementary embedding $i : V \to M$ such that κ is the first ordinal α such that $\alpha < i(\alpha)$. In the following two exercises you prove that the converse also holds.

5. Assume that $i : V \to M$ is an elementary embedding which is not the identity mapping, and M is transitive.

(a) There is an ordinal α such that $i(\alpha) > \alpha$.
(b) For each n, $i(n) = n$, and $i(\omega) = \omega$.

6. Assume that $i : V \to M$ is an elementary embedding which is not the identity mapping, and M is transitive. Then $\kappa := \min\{\alpha : \alpha < i(\alpha)\}$ is a measurable cardinal.

Hint for 1:(a) By E-recursion, define $\pi(x) = \{\pi(y) : yEx\}$, and $M = \pi[P]$.
(b) Use E-induction.

*Hint for 3(b):*Set-like: Fix $[g]$. If $[f]E[g]$, let $h(\alpha) = f(\alpha)$ when $f(\alpha) \in g(\alpha)$, and $h(\alpha) = 0$ otherwise. $[h] = [f]$. $\{[f] : [f]E[g]\} = \{[h] : h \in^* g,\ (\forall \alpha < \kappa)\ h(\alpha) \in g(\alpha) \cup \{0\}\}$.
Well-founded: If $\dots [f_2]E[f_1]E[f_0]$, take $\alpha \in \bigcap_{n\in\mathbb{N}} \{\alpha < \kappa : f_{n+1}(\alpha) \in f_n(\alpha)\}$.

*Hint for 4:*Induction on the structure of φ.

*Hint for 5(a):*For each x, $i(\mathrm{rank}(x)) = \mathrm{rank}(i(x))$. Assume that for each α, $i(\alpha) = \alpha$. Then $\mathrm{rank}(i(x)) = \mathrm{rank}(x)$ for all x. By induction on the rank, $i(x) = x$ for all x. ($y \in i(x)$ implies $y = i(y)$.)

*Hint for 5(b):*Induction on n for the first assertion; definition of ω for the second. ($V \models \varphi(\omega)$, where $\varphi(x) = 0 \in x \wedge \forall y \in x(y \neq 0 \to \exists z \in x(y = z \cup \{z\}))$. $i[V] \models \varphi(i(\omega))$. $\varphi(i(\omega))$ holds.)

Hint for 6:$\kappa > \omega$. Let
$$\mathcal{U} = \{X \subseteq \kappa : \kappa \in i(X)\}.$$
$\kappa \in \mathcal{U}$. $\emptyset \notin \mathcal{U}$.
$X \subseteq Y \to i(X) \subseteq i(Y)$ (express $X \subseteq Y$ as a formula).
$i(\kappa \setminus X) = i(\kappa) \setminus i(X)$.
For all $\alpha < \kappa$, $i(\{\alpha\}) = \{i(\alpha)\} = \{\alpha\}$.
κ-completeness: Fix $\gamma < \kappa$ and $f : \gamma \to \mathcal{U}$. You must show that $\bigcap_{\alpha<\gamma} f(\alpha) \in \mathcal{U}$:
For each $\alpha < \gamma$, if $X = f(\alpha)$, then $i(X) = i(f)(i(\alpha))$. (Express $X = f(\alpha)$ as a formula $\varphi(X, f, \alpha)$.)
As $\alpha < \gamma$, $i(f(\alpha)) = i(X) = i(f)(\alpha)$.
Express "$A = \bigcap_{\alpha<\gamma} f(\alpha)$" as a formula $\varphi(A, \gamma, f)$.
$$i(A) = \bigcap_{\alpha<i(\gamma)} i(f)(\alpha) = \bigcap_{\alpha<\gamma} i(f(\alpha)) \ni \kappa.$$

Good luck!

Department Mathematics, Bar-Ilan University; and Department Mathematics, Weizmann Institute of Science
E-mail address: tsaban@math.biu.ac.il
URL: http://www.cs.biu.ac.il/~tsaban