A TASTE OF SET THEORY: EXERCISE 11

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1. Assume that μ is a (nontrivial probability) measure on a set X, $\alpha < \aleph_1$, and $(A_\beta : \beta < \alpha)$ is an α-sequence of μ-measurable sets. Prove:

(1) If the sequence is increasing with α (i.e., $A_\beta \subseteq A_\gamma$ for all $\beta < \gamma$), then $\mu(\bigcup_{\beta<\alpha} A_\alpha) = \sup\{\mu(A_\beta) : \beta < \alpha\}$.
(2) If the sequence is decreasing with α (i.e., $A_\beta \supseteq A_\gamma$ for all $\beta < \gamma$), then $\mu(\bigcap_{\beta<\alpha} A_\alpha) = \inf\{\mu(A_\beta) : \beta < \alpha\}$. (In particular, if $\mu(A_\beta) > 1/2$ for all $\beta < \alpha$, then $\mu(\bigcap_{\beta<\alpha} A_\alpha) \geq 1/2$.)

Hint. For (1), consider the sets $A_{\beta+1} \setminus A_\beta$. For (2), use (1).

2. Prove:

(1) For each transitive set A, $A \subseteq P(A)$.
(2) For each transitive set A, $P(A)$ is transitive.
(3) For each α, V_α is transitive.
(4) For all $\alpha \leq \beta$, $V_\alpha \subseteq V_\beta$.

Definition. For each x, define the *rank of* x to be $\mathrm{rank}(x) = \min\{\alpha : x \in V_{\alpha+1}\}$.

3. Prove:

(1) For each α, $V_\alpha = \{x : \mathrm{rank}(x) < \alpha\}$.
(2) For all x, y: If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.
(3) For each y, $\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 : x \in y\}$.
(4) For each α, $\mathrm{rank}(\alpha) = \alpha$.
(5) For each α, $V_\alpha \cap ON = \alpha$.

4. Prove:

(1) For each x, the ranks of $\bigcup x, P(x), \{x\}$, are all less than $\mathrm{rank}(x) + \omega$ (ordinal addition).
(2) For all x, y, the ranks of $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), {}^y x$, are all less than $\max\{\mathrm{rank}(x), \mathrm{rank}(y)\} + \omega$.
(3) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in V_{\omega+\omega}$.

Good luck!

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