A TASTE OF SET THEORY: EXERCISE 10

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Definition. A subset C of a partially ordered set P is *cofinal* (in P) if for each $p \in P$, there is $d \in D$ such that $p \leq d$. The *cofinality* of P, denoted $\operatorname{cof}(P, \leq)$, is the minimal cardinality of a cofinal subset of P.

For example, for an ordinal α, $\operatorname{cof}(\alpha, \leq)$ is just $\operatorname{cf}(\alpha)$.

1. Consider $\mathbb{N}^\mathbb{N}$ with the partial order defined by $f \leq g$ if and only $f(n) \leq g(n)$ for all n. Prove that $\operatorname{cof}(\mathbb{N}^\mathbb{N}, \leq) = \mathfrak{d}$.

2. Prove:

(1) For each singular κ, $\operatorname{cof}([\kappa]^{\operatorname{cf}(\kappa)}, \subseteq) > \kappa$.
[*Hint.* The proof of König's Lemma.]
(2) Show that (1) implies König's Lemma for singular κ.
(3) What is $\operatorname{cof}([\kappa]^{\operatorname{cf}(\kappa)}, \subseteq)$ when κ is regular?

Definition. A nonempty family $\mathcal{S} \subseteq [\mathbb{N}]^{\aleph_0}$ is a *superfilter* (on $\mathbb{N}$) if for all $A, B \subseteq \mathbb{N}$:

(a) If $A \in \mathcal{S}$ and $B \supseteq A$, then $B \in \mathcal{S}$.
(b) If $A \cup B \in \mathcal{S}$, then $A \in \mathcal{S}$ or $B \in \mathcal{S}$.

3. Prove:

(1) There is a superfilter on $\mathbb{N}$ which is not a filter.
(2) The definition of superfilter does not change if we assume that A, B are disjoint in (b).
(3) If we make the change suggested in (2), and in addition we replace there *or* by *exclusive or*, we obtain a characterization of (nonprincipal) ultrafilter on $\mathbb{N}$.

Good luck!

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