I-FAVORABLE SPACES AND INVERSE SYSTEMS.

ANDRZEJ KUCHARSKI AND SZYMON PLEWIK

Abstract. It is showed that any compact space X is I-favorable if, and only if X can be representing as a limit of σ-complete inverse system of compact metrizable spaces with skeletal bonding maps.

[Compact openly generated spaces.]

In several papers [13], [14] and [15], E.V. Shchepin considered a few classes of compact spaces. Among others, he introduced the class of compact openly generated spaces. A compact space X is called *openly generated*, whenever X is a limit of a σ-complete inverse system of metrizable spaces with open bonding maps. Since Ivanov's result [9]: *A compact space X is openly generated if, and only if its superextension is a Dugundji space*; Shchepin established that the classes of openly generated compact spaces and of κ-metrizable spaces are the same, see Theorem 21 in [15].

[Limits of inverse systems with skeletal bonding maps.]

We consider the class of topological (compact) spaces which widen openly generated spaces. Our's class consists of limits of σ-complete inverse systems of compact metrizable spaces with skeletal bonding maps. It occurs that compact spaces from this class are the same as I-favorable spaces, Theorems 1 and 3. Any continuous and open map is skeletal, hence every compact openly generated space has to be I-favorable.

[The converse is not true.]

There are extremally disconnected and I-favorable spaces. For instance βN, i.e. the Čech-Stone compactification of positive integers with the discrete topology, is I-favorable and extremally disconnected, but βN is not openly generated, since any compact extremally disconnected and openly generated space has to be discrete, see Theorem 11 in [13].

[Connections with Boolean algebras.]

A Boolean algebra $\mathbb{B}$ is semi-Cohen (regularly filtered) if, and only if $[\mathbb{B}]^\omega$ has a closed unbounded set of countable regular subalgebras, in other words $[\mathbb{B}]^\omega$ contains a club filter. The Stone space of a semi-Cohen algebras is I-favorable, compare [2] and [7]. Thus our's results develop facts on semi-Cohem algebras, compare Theorem 4.3 in [2].

[Sketch of results.]

We modify quotient topologies and quotient maps. We introduce and consider indentification topologies and identification maps. We extract conditions $(\mathcal{W}^{\mathcal{T}})$ from properties of cozero sets. Frink's theorem is used to show that $X/\mathcal{T}$ is completely regular, whenever $\mathcal{T}$ is a ring of subsets of X and each $W \in \mathcal{T}$ fulfills $(\mathcal{W}^{\mathcal{T}})$. We introduce skeletal families to modify the essensial property of club filters, in particular the condition $(\mathcal{S})$. Representations of I-favorable compact spaces as limits of σ-complete inverse systems of compact metrizable spaces with skeletal bonding maps are our's main result.

[Main results.]

Representations of I-favorable compact spaces as limits of σ-complete inverse systems of compact metrizable spaces with skeletal bonding maps are our's main result. These results are similar to Shchepin's theory of openly generated spaces.

Theorem 1. *If X is a I-favorable compact space, then*

$$X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\},$$

where $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, all spaces X_σ are compact and metrizable, and all bonding maps π^σ_ϱ are skeletal.

Theorem 2. *Let $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ be a σ-complete inverse system with all bonding maps skeletal. If all spaces X_σ are metrizable and separable, then the limit*

$$X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$$

is I-favorable.

[Inverse systems: σ-complete.]

A directed set Σ is said to be σ-complete if any countable chain of its elements has least upper bound in Σ. An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a σ-complete, whenever Σ is σ-complete and for every chain $\{\sigma_n : n \in \omega\} \subseteq \Sigma$, such that $\sigma = \sup\{\sigma_n : n \in \omega\} \in \Sigma$, there holds

$$X_\sigma = \varprojlim\{X_{\sigma_n}, \pi^{\sigma_{n+1}}_{\sigma_n}\},$$

compare E.V. Shchepin [15] (1981). For other details see R. Engelking [5] (1977), pages 135 - 144.

[Skeletal functions.]

A continuous function is called *skeletal* whenever for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior. If X is a compact space and Y Hausdorff, then a continuous $f : X \to Y$ is skeletal if, and only if $\operatorname{Int} f[U] \neq \emptyset$, for every open $U \neq \emptyset$. In the literature one can find equivalent notions: *almost-open or semi-open*; see A. Arhangelskii [1] (1962), and H. Herrlich and G.E. Strecker [8] (1968). We call such maps skeletal following J. Mioduszewski and L. Rudolf [11] (1969).

[Remainder of βN.]

There exist topological spaces with no skeletal map onto a dense in itself metrizable space. For example, the remainder of the Čech-Stone compactification βN. In fact, any space such that every decreasing sequence of open nonemty subsets has the intersection with non-empty interior has no skeletal map onto a dense in itself and Hausdorff space.

[Souslin lines, density topologies.]

If I is a compact segment of connected Souslin line and X is metrizable, then each skeletal map $f : I \to X$ has to be constant. Indeed, let Q be a countable and dense subset of $f[I] \subseteq X$. Suppose a skeletal map $f : I \to X$ is non constant. Then the preimage $f^{-1}(Q)$ is nowhere dense in I as countable union of nowhere dense subset of a Souslin line. So, for each open set $V \subseteq I \setminus f^{-1}(Q)$ there holds $\operatorname{Int} f[V] = \emptyset$, a contradiction. Another example is a regular Baire space X with a category measure μ such that $\mu(X) = 1$, for a definition of this space see [12, pp. 86 - 91].

[Independent example.]

In [3] A. Błaszczyk and S. Shelah considered separable extremally disconnected spaces without skeletal map onto a dense in itself metrizable space. Theirs result is stated in terms of Boolean algebra: *There is a nowhere dense ultrafilter on ω if, and only if there is a complete, atomless, σ-centered Boolean algebra which contains no regular, atomless, countable subalgebra.*

[Open-open game.]

Players are playing with a topological space X in the open-open game. Player I chooses a non-empty open subset $A_0 \subseteq X$ at the beginning. Then Player II chooses a non-empty open subsets $B_0 \subseteq A_0$. Player I chooses a non-empty open subset $A_n \subseteq X$ at the n-th round, and then Player II chooses a non-empty open subset $B_n \subseteq A_n$. Player I wins, whenever the union $B_0 \cup B_1 \cup \ldots \subseteq X$ is dense. One can assume that Player II wins for other cases. The space X is called I-*favorable* whenever Player I can be insured that he wins no matter how Player II plays. In other words, Player I has a winning strategy. For more details about the open-open game see P. Daniels, K. Kunen and H. Zhou [4] (1994).

[Club filter.]

Fix a π-base $\mathcal{Q}$ for a space X. Following [4], compare [10], any family $\mathcal{C} \subset [\mathcal{Q}]^\omega$ is called *a club filter* whenever:

The family $\mathcal{C}$ is closed under ω-chains with respect to inclusion;

For any countable subfamily $\mathcal{A} \subseteq \mathcal{Q}$, where $\mathcal{Q}$ is the $\pi-$base fixed above, there exists $\mathcal{P} \in \mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{P}$; and

($\mathcal{S}$). *For any non-empty open set V and each $\mathcal{P} \in \mathcal{C}$ there is $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then U meets V, i.e. $U \cap V \neq \emptyset$.*

[Known characterization.]

There holds, see [4] Theorem 1.6, compare [10] Lemmas 3 and 4: *A topological space has a club filter if, and only if it is I-favorable.* In the next part we modify a little the definition of club filters.

[Skeletal families.]

Any $\mathcal{P}$ closed under a winning strategy for Player I fulfills $(\mathcal{S})$, hence the condition $(\mathcal{S})$ gives reasons to look into I-favorable spaces with respect to skeletal families. A family $\mathcal{P}$ of open subsets of a space X is called *skeletal family*, whenever for every non-empty open set $V \subseteq X$ there exists $W \in \mathcal{P}$ such that $U \subseteq W$ and $\emptyset \neq U \in \mathcal{P}$ implies $U \cap V \neq \emptyset$.

Lemma 3. *Let $f : X \to Y$ be a continuous function and let $\mathcal{B}$ be a π-base for Y. The family $\{f^{-1}(V) : V \in \mathcal{B}\}$ is skeletal if, and only if f is a skeletal map.*

[Identification maps, topologies.]

Let $\mathcal{P}$ be a family of subsets of X. We say that $y \in [x]_{\mathcal{P}}$, whenever for every $V \in \mathcal{P}$ there holds $x \in V$ if, and only if $y \in V$. The family of all classes $[x]_{\mathcal{P}}$ is denoted $X/\mathcal{P}$. Put $q(x) = [x]_{\mathcal{P}}$. The function $q : X \to X/\mathcal{P}$ is called an *identification map.* The minimal topology on $X/\mathcal{P}$, containing images $q[V] = \{[x]_{\mathcal{P}} : x \in V\}$ for $V \in \mathcal{P}$, is called an *identification topology.* If X is a compact space and $X/\mathcal{P}$ is Hausdorff, then the identification map $q : X \to X/\mathcal{P}$ is the natural quotient mapping. Also, the identification topology coincides with the quotient topology, compare [5] p. 124.

Corollary 4. *Let $\mathcal{P}$ be a family of open subsets of X. If $X = \bigcup \mathcal{P}$ and $\mathcal{P}$ is closed under finite intersections, then the identification map $q : X \to X/\mathcal{P}$ is continuous. Moreover, the family $\{[V] : V \in \mathcal{P}\}$ is a base for the identification topology on $X/\mathcal{P}$.*

[Property of cozero sets.]

($\mathcal{W}^{\mathcal{T}}$): *There exist sets* $\{U_n : n \in \omega\} \subseteq \mathcal{T}$ *and* $\{V_n : n \in \omega\} \subseteq \mathcal{T}$ *such that* $U_k \subseteq (X \setminus V_k) \subseteq U_{k+1}$, *for any* $k \in \omega$, *and* $\bigcup\{U_n : n \in \omega\} = W$.

[T_2 and T_3 spaces.]

Lemma 5. *If* $\mathcal{T}$ *is a family of sets such that each* $W \in \mathcal{T}$ *fulfills* ($\mathcal{W}^{\mathcal{T}}$), *then* $X/\mathcal{T}$ *is Hausdorff.*

Lemma 6. *Let* $\mathcal{T}$ *be a family of sets such that each* $W \in \mathcal{T}$ *fulfills* ($\mathcal{W}^{\mathcal{T}}$). *If* $\mathcal{T}$ *is closed under finite intersections, then* $X/\mathcal{T}$ *is regular.*

[Completely regular spaces.]

To get that $X/\mathcal{P}$ is completely regular we should apply the Frink's theorem, see [6] or [5] p. 72. Recall a reformulation of this theorem.

Theorem [O. Frink (1964)]. *A* T_1*-space* X *is completely regular if, and only if there exists a base* $\mathcal{B}$ *satisfying*:

(1) *If* $x \in U \in \mathcal{B}$, *then there exists* $V \in \mathcal{B}$ *such that* $x \notin V$ *and* $U \cup V = X$;

(2) *If* $U, V \in \mathcal{B}$ *and* $U \cup V = X$, *then there exists disjoint sets* $M, N \in \mathcal{B}$ *such that* $X \setminus U \subseteq M$ *and* $X \setminus V \subseteq N$. ☕

Theorem 7. *If* $\mathcal{T}$ *be a ring of subsets of* X *such that each* $W \in \mathcal{T}$ *fulfills the conditions* ($\mathcal{W}^{\mathcal{T}}$), *then* $X/\mathcal{T}$ *is a completely regular space.*

[Identification maps are skeletal.]

Suppose that, each $W \in \mathcal{T}$ fulfills $(\mathcal{W}^{\mathcal{T}})$. If $\mathcal{T}$ is finite, then $X/\mathcal{T}$ is discrete, as a finite Hausdorff space. Whenever $\mathcal{T}$ is countable and closed under finite intersections, then $X/\mathcal{T}$ is a Hausdorff and regular space with a countable base. Then it is a metrizable separable space.

Theorem 8. *If a ring $\mathcal{P}$ of open subsets of X is closed under a winning strategy and each $W \in \mathcal{P}$ fulfills $(\mathcal{W}^{\mathcal{P}})$, then $X/\mathcal{P}$ with the identification topology is completely regular and the identification map $q : X \to X/\mathcal{P}$ is skeletal.*

[$\mathcal{T}$-clubs.]

We introduce $\mathcal{T}$-clubs, i.e. club filters with some additional properties. Consider a collection $\mathcal{C} = \{\mathcal{P}(\mathcal{Q}) : \mathcal{Q} \in [\mathcal{T}]^{\omega}\}$ such that each $\mathcal{P} \in \mathcal{C}$ is countable and closed under a winning strategy for Player I and all strategies σ_k^*: i.e. any $W \in \mathcal{P}$ fulfills $(\mathcal{W}^{\mathcal{P}})$; and closed under finite intersections and finite unions. Then, the family $\mathcal{C}$ is called $\mathcal{T}$-*club*.

[Properties of $\mathcal{T}$-clubs.]

Any $\mathcal{T}$-*club* $\mathcal{C}$ is closed under ω-chains with respect to the inclusion and each $\mathcal{P} \in \mathcal{C}$ is a ring of sets. Each $\mathcal{P} \in \mathcal{C}$ fulfills $(\mathcal{W}^{\mathcal{P}})$. Additionally, any $\mathcal{P} \in \mathcal{C}$ is closed under a winning strategy for Player I, hence for each $\mathcal{P} \in \mathcal{C}$ the identification map $q_{\mathcal{P}} : X \to X/\mathcal{P}$ is skeletal and onto a metrizable separable space. These properties suffice to build an inverse system with skeletal bonding maps onto metrizable separable spaces.

[$\mathcal{T}$-clubs and inverse systems.]

Any $\mathcal{T}$-club $\mathcal{C}$ is directed by the inclusion. For each $\mathcal{P} \in \mathcal{C}$ it is assigned the identification space $X/\mathcal{P}$ and the skeletal function $q_{\mathcal{P}} : X \to X/\mathcal{P}$. If $\mathcal{P}, \mathcal{R} \in \mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{R}$, then put $q^{\mathcal{R}}_{\mathcal{P}}([x]_{\mathcal{R}}) = [x]_{\mathcal{P}}$. Thus, we have defined the inverse system $\{X/\mathcal{R}, q^{\mathcal{R}}_{\mathcal{P}}, \mathcal{C}\}$. In this inverse system spaces $X/\mathcal{R}$ are metrizable and separable, bonding maps $q^{\mathcal{R}}_{\mathcal{P}}$ are skeletal and the directed set $\mathcal{C}$ is σ-complete.

[Completely regular cases.]

Lemma 9. *Let X be a I-favorable completely regular space. If $\mathcal{C}$ is a $\mathcal{T}$-club, then $Y = \varprojlim\{X/\mathcal{R}, q^{\mathcal{R}}_{\mathcal{P}}, \mathcal{C}\}$ contains a dense subspace homeomorphic with X.*

Back to the main results.

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