III Workshop on
Coverings, Selections and
Games in Topology
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The centennial of Djuro Kurepa

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Djuro R. Kurepa: August 16, 1907 (Majske Poljane) – November 2, 1993 (Beograd)

University of Zagreb (Faculty of Mathematics and Physics) in 1931.

Doctoral dissertation: Paris, December 19, 1935; ("Ensembles ordonnés et ramifiés", 6 + 138 + 2; *Ensembles ordonnés et ramifiés, Publ. Math. Univ. Belgrade 4 (1935), 1-138*); Committee: Paul Montel, president, Maurice Fréchet, supervisor and reporter, Arnauld Denjoy.

University of Zagreb (1938–1965);

University of Belgrade (1965–1877);

Professor Emeritus (1977–1993)

About 300 scientific works (the first 1933; the last 1993)

The author or coauthor of 147 books with 53 first editions.

"Set Theory", 1951

"Higher Algebra" (Zagreb, 1965; Beograd, 1971) in two volumes, 1519 pages.

Advisor of 49 master theses and 42 doctoral theses.

Cofounder and Editor-in-Chief of the journals Glasnik Mat. Fiz. Astr.(Zagreb, 1945) and Math. Balkanica (Beograd, 1971-1981; since 1981 until his death Kurepa was a honorary editor).

Attended:

all International Mathemathical Congresses from 1952 until 1982 (at the Edinburgh Congress he was an invited speaker on "Some principles of mathematical education"),

all Balkan Mathematical Congresses (1967-1983),

all 8 Yugoslav Mathematical Congresses from 1949 to 1985,

three Congresses on Methodology of Sciences (Jerusalem, 1964; Hannover, 1979; Salzburg, 1983),

two International Congresses on Philosophy (Brussels,1953; Venezia, 1958).

Head of the Mathematical Institute in Zagreb (1943–1965)

Head of the Mathematical Institute in Belgrade (1969–1976).

President of the Union of Societies of Mathematicians, Physicists and Astronomers of Yugoslavia (1955-1960)

President of the Balkan Union of Mathematicians (1977-1983)

Vice-president of the ICMI (1952–1962)

Honorary member of the Nikola Tesla Memorial Society for Canada and USA (1982).

Corresponding member of the Yugoslav Academy of Sciences and Arts in Zagreb (1952)

Member of the Serbian Academy of Sciences and Arts in Belgrade (1988)

Member of the Academy of Sciences and Arts of Bosnia and Hertzegovina (1984)

Member of the Academy of Sciences of Macedonia.

AVNOJ Prize (1976) - the greatest scientific and cultural award in former Yugoslavia (only one mathematician yet was awarded this prize: I. Vidav, Ljubljana)

The Order of work with red flag (1965),

Order for merits for people with golden star (1975)

The Bernard Bolzano Medallion, Praha (1981)

SET THEORY

1. Souslin problem

SH: Every ccc compact LOTS is separable

SL: counterexample of SH

GSH: For every LOTS X, $c(X) = d(X)$ (Kurepa's thesis)

Kurepa, 1935: SL exists iff there exists a Suslin tree (i.e. SH $\Leftrightarrow$ there is no a Suslin tree)

[This result is sometimes attributed to W. Miller (1943)]

2. Kurepa tree (1935, also 1942):

an ω_1-tree with more than ω_1 cofinal branches

Kurepa Hypothesis KH: Kurepa trees exist

F. Rowbottom (1965); L. Bukovsky (1966): If ω_1 is inaccessible in L and if no ordinal between ω_1 and ω_2 belongs to L, then KH is true

D.H. Stewart (1966): there exists a (standard transitive) model of ZFC in which there is a Kurepa tree

J.H. Silver: there is a model of ZFC such that $2^{\omega_1} > \omega_2$ holds and there exists a Kurepa tree having exactly ω_2 cofinal branches

J.H. Silver (1966): There is a model of ZFC in which Kurepa trees do not exist

R.M. Solovay: Axiom of constructibility $V = L$ implies the existence of Kurepa trees

K. Devlin (1981): (relative) consistency of MA$+$ $\neg$KH with ZFC.

3. Kurepa line (= KH)

Kurepa family: a family $\mathcal{K}$ of subsets of ω_1 such that: (i) $|\mathcal{K}| > \omega_1$; (ii) $\{K \in \mathcal{K} : K \cap \alpha \neq \emptyset\}$ is countable for each countable α

Kurepa continuum: a linearly ordered continuum K of weight ω_1 having more than ω_1 points of uncountable character and such that the closure of every countable subset of K has a countable base

Kurepa type: an order type ϕ having the following three properties: (i) $|\phi| > \omega_1$; (ii) $d(\phi) = \omega_1$; (iii) ϕ does not contain uncountable real type

Kurepa τ-partially ordered set

KH (the existence of a Kurepa line) and the existence of a Kurepa family are equivalent

4. Weak Kurepa tree: A tree of height ω_1 and cardinality ω_1 with more than ω_1 cofinal branches

The weak Kurepa Hypothesis wKH: there exists a weak Kurepa tree

W.J. Mitchel (1972): (relative) consistency of $\neg$wKH

CH implies wKH and that MA$+\neg$wKH is stronger than MA $+ \neg$CH ($\neg$wKH is a strengthening of $\neg$CH);

PFA implies MA $+ \neg$wKH

PFA $\Rightarrow$ wKH is false

S. Todorčević (1981): MA$+\neg$wKH is consistent with ZFC

5. T a tree; $ch(T)$ (resp. $ach(T)$) denotes the supremum of cardinalities of all chains (resp., antichains) in T; $bT = \sup\{ch(T), ach(T)\}$

Kurepa (1935): For every tree T, $|T| \leq 2^{bT}$; in fact, it is easy to check that for every tree T, $|T| \leq (chT)^{+}(achT)$

Kurepa's Ramification Hypothesis (or Rectangle Hypothesis) RH (1936): for every tree T we have $|T| \leq bT$

SH is true iff for every tree T, $bT \leq \omega$ implies $|T| \leq \omega$, i.e. SH $\Leftrightarrow$ in every uncountable tree T there exists an uncountable chain or antichain

In 1989, Kurepa introduced a class of mappings called almost strictly increasing and proved an equivalent of RH in terms of these mappings.

6. Kurepa antichain principle KAP**:** AC is equivalent to the conjuction of (1) every set can be linearly ordered, and (2) every partially ordered set has a maximal antichain.

7. The problem of existence of strictly increasing real-valued functions on trees and partially ordered sets

1940, 1941: SH is equivalent to the statement: if T is a tree with $bT \leq \omega$, then there exists a strictly increasing function $f : T \mapsto \mathbb{R}$

1942: A partially ordered set P admits a strictly increasing function into the set of rationals iff it can be decomposed into at most countably many antichains (and that such a function from P into $\mathbb{R}$ exists if every well-ordered subset of P is at most countable)

1954: For a partially ordered set P denote by σP the set of all bounded well-ordered subsets of P ordered by $A \leq B$ iff A is an initial segment of B. Then σP is a tree.

$\sigma\mathbb{Q}$ (resp. $\sigma\mathbb{R}$) does not admit a strictly increasing mapping into $\mathbb{Q}$ (resp. $\mathbb{R}$)

1989: Kurepa generalized some previously mentioned results to any aleph

8. Kurepa (1953): Conjecture that $c = 2^\omega$ can be any uncountable aleph of cofinality $> \omega$.

Surprisingly, this conjecture was true (R. Cohen and for a generalization to higher cardinality, for regular cardinals, W.E. Easton 1970). For singular cardinals the problem remains open.

9. Partition Calculus:

Very important, but unfortunately not recognized enough, are Kurepa's contributions to Partition Calculus.

F.P. Ramsey (1930), W. Sierpinski (1933), Kurepa, P. Erdös, Erdös-R. Rado, A. Hajnal and others.

1937 (without proof), 1939 (with a proof): If X is partially ordered, then

$$|X| \leq (2k_s(X))^{k_0(X)};$$

$k_s(X) = \sup\{|Y| : Y \text{ is an antichain in } X\}$,

$k_0(X) = \sup\{|Y| : Y \subset X \text{ is WO or conversely WO}\}$.

1939 and 1952/53, again 1959: For every graph (G, ρ) we have

$$|G| \leq (2k_s(G))^{k_c(G)};$$

$k_c(G) = \text{sup}\{|X| : X \subset G \text{ is connected}\}$,

$k_s(G) = \text{sup}\{|Y| : Y \subset G$ is anticonnected, i.e. $a \neq b$ in G implies a nonρ $b\}$.

S. Todorčević (Partition Problems in Topology, 1989) uses:

Erdös-Hajnal-Kurepa-Rado theorem for

$$(exp_{s-1}\tau)^+ \rightarrow (\tau^+)^s_\tau$$

known usually as Erdös-Rado theorem

Erdös-Kurepa-Rado-Sierpinski theorem for

$$(\tau^+, \tau^+)^2 \rightarrow ((2^\tau)^+, \tau^\tau)^2.$$

I. Juhász (Cardinal Functions in Topology, 1971) gives Kurepa's result from 1959:

$$2^\tau \nrightarrow (\tau^+, \tau^+)^2$$

(as a generalization of the result of Sierpinski

$$2^\omega \nrightarrow (\omega_1, \omega_1)^2.)$$

TOPOLOGY

1. Kurepa (1934): **Pseudodistancial spaces** (known today usually as ω_μ-metrizable or lineraly uniformizable spaces).

M. Fréchet (1945): linearly uniformizable spaces

J. Colmez (1947): this class coincides with the class of pseudodistancial spaces.

R. Sikorski (1950): ω_μ-metric spaces

F.W. Stevenson, W.J. Thron (1969): this class is the class of linearly uniformizable spaces.

1936: For every initial ordinal ω_α the class $(\mathcal{D}_\alpha)$ as special pseudodistancial spaces

Kurepa proved: D_0 is exactly the class of metric spaces

Kurepa (1963): For every regular uncountable cardinal τ there exists a τ-metrizable space which is not orderable

Consequence: If τ is a regular uncountable cardinal, then every dense-in-itself τ-metrizable space is orderable

[The last result was rediscovered by R. Frankiewicz and W. Kulpa (1979)]

For topological groups by P. Nyikos & H.-C. Reichel (1975): ω_μ-metrizable non metrizable topological group is orderable.

2. Kurepa (1936): R-**spaces** (= "spaces with ramified bases") as T_1-spaces having a base which is a tree (with respect to $\supset$).

These spaces form a class that is wider than the class of linearly uniformizable spaces;

P. Papić (1954): Every pseudodistancial non-metrizable space is an R-space.

These spaces were redefined a few times by some authors and under different names.

A.F. Monna (1950): Non-archimedean spaces (which are in fact R-spaces).

Every ω_μ-metrizable space is non-archimedean for $\mu > 0$ (for $\mu = 0$ X is non-archimedean iff $dim\, X = 0$ by a result of de Groot).

Non-archimedean metrics are called ultra-metrics and were known to Hausdoff (1914)

A.V. Arhangel'skii (1962, 1963): spaces with a base of rank 1 which are precisely R-spaces.

P. Papić (1954), Kurepa (1956): an R-space is metrizable if and only if it has a base which is a tree of countable height.

A.V. Arhangel'skii, V.V. Filippov (1973): Every R-spase which is also a p-space is metrizable.

P.Papić (1963): an R-space is orderable if and only if it has a base which is a tree having no compact elements in levels whose order type is a limit ordinal.

This condition is equivalent to the following Kurepa's condition (δN_2) from 1956: every member in a level whose order type is a limit ordinal is a union of open and closed pairwise disjoint subsets of X.

3. The notion of **cellularity** of a space was introduced by Kurepa in his thesis in 1935.

Kurepa's significant results on the cellularity:

1935: $c(X)$ is accessible for metric spaces X and for linearly ordered spaces always but in the case when $c(X)$ is an inaccessible cardinal

1953: $s(X)$ is accessible whenever X is a metric space or a linearly ordered space unless $s(X)$ is an inaccessible cardinal.

1950: If S is the Suslin line, then $c(S \times S) > c(S)$ (more precisely, $c(S \times S) = c(S)^{+}$).

1962: If τ is a cardinal and if $\{X_\lambda : \lambda \in \Lambda\}$ is a family of topological spaces such that $c(X_\lambda) \leq \tau$ for all $\lambda \in \Lambda$, then $c(\prod\{X_\lambda : \lambda \in \Lambda\}) \leq 2^\tau$.

4. Cardinal invariants on LOTS':

(1) $d(X) \leq c(X)^+$ (1935)

(2) $hd(X) = d(X)$ and $hl(X) = c(X)$ (1939, 1945)

The first of these two results was obtained later by L. Skula in 1965, and the second one by S. Mardešić & P. Papić in 1962 for compact case and by R. Bennett & D. Lutzer in 1969 for any LOTS.

(3) $c(X) = s(X)$.

(4) $d(X) \leq c(X^2)$ (1952)

S. Todorčević: $hd(X) \leq \min\{c(X)^+, c(X^2)\}$.

This result gives the following remarkable consequence: If X is a linearly ordered continuum and if for some $n > 1$, $c(X^n) \leq \omega$, then X is similar to a segment of the real line; so the only exception is the case $n = 1$ (SH).

NUMBER THEORY

1971 (**left factorial**): $!n = 0! + 1! + ... + (n-1)!$

In this paper and a few papers published in the subsequent years Kurepa stated the following

Conjecture (Problem B44 in: Richard K. Guy, Unsolved problems in Number Theory, Springer-Verlag, 1980): for every $n \in \mathbb{N}$, the greatest common divisor for $n!$ and $!n$ is equal to 2.

By the use of computers this hypothesis was checked for all $n < 10^6$ and for such numbers it is true

(D. Slavić for $n < 10^3$; Wagstaff for $n < 50000$; Ž. Mijajlović for $n < 317000$; Ž. Mijajlović and G. Gogić for $n < 10^6$).

In 2003: YES

1973: (K-**function**)

Using

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx = (n-1)!,\ n \in \mathbb{N},$$

one obtains

$$!n = \int_0^\infty e^{-x} \frac{x^n - 1}{x - 1} dx.$$

Kurepa defined the same formula for the complex domain:

$$!z = \int_0^\infty e^{-x} \frac{x^z - 1}{x - 1} dx,\ Re(z) > 0.$$