BASE MATRIX TREE FOR DOUGHNUTS

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Notations.

For a set x, let $|x|$ denote its cardinality and let $[\omega]^\omega = \{X \subseteq \omega : |X| = \omega\}$. If $S \subseteq [\omega]^\omega$, then put $S^* = \{x \triangle P : P \in S$ and $x \in [\omega]^{<\omega}\}$. (...)

Doughnut property.

In [5] C. Di Prisco and J. Henle introduced the so-called doughnut property: For $A \subseteq B \subseteq \omega$ with $B \setminus A \in [\omega]^\omega$, the set

$$< A, B >= \{X \in [\omega]^\omega : A \subseteq X \subseteq B\}$$

is called *doughnut*. A subset $S \subseteq [\omega]^\omega$ has the *doughnut property*, whenever it contains or is disjoint from some doughnut.

Fisrt palce finite or infinite.

If $< A, B >$ and $< C, D >$ are doughnuts, then the intersection $< A, B > \cap < C, D >$ is finite or is a doughnut. But, for $a, c \in [\omega]^{<\omega}$ and $B, D \in [\omega]^\omega$ the intersection $< a, B > \cap < c, D >$ is empty or has the cardinality continuum.

CD sets and ND sets.

Following L. Halbeisen [8] we call a subset $S \subseteq [\omega]^\omega$ *completely doughnut*, briefly S is a CD set, whenever for each doughnut $< A, B >$ there exists a doughnut $< C, D > \subseteq < A, B >$ such that

$$< C, D > \subseteq S \text{ or } < C, D > \cap S = \emptyset.$$

But, whenever always holds $< C, D > \cap S = \emptyset$, then S is called *doughnut null*, briefly S is a ND set.

σ-field, σ- ideal

Any subset of a ND set is a ND and CD set. Also, each complement of a CD set is a CD set, too. The family of all CD sets is a σ-field and the family of all ND sets is a σ- ideal, see facts 1.3, 1.5 and 1.6 in Halbeisen [8].

Comparison with completely Ramsey sets.

Notions of CD set and ND set are counterpars of completely Ramsey set, i.e. CR set, and nowhere Ramsey set, i.e. NR set. Indeed, CR sets was introduced by J. Sliver [12]. F. Galvin and K. Prikry showed that all CR sets form a σ-field, see Lemmas 6 and 10 in [7]. Thus, the definition of a NR set is natural: Any CR set S such that each subset $P \subseteq S$ is a CR set is called NR set.

Families of CR and CD sets are diffenrent.

One can check that any Berstein subset restricted to a doughnut $< A, B >$, where $A \in [\omega]^\omega$, is NR and not CD. If $\mathcal{H}$ is a base matrix tree, for exact definition of $\mathcal{H}$ see the base matrix lemma, i.e. Lemma 2.11 in [2], then $(\bigcup \mathcal{H})^*$ is not a CR set and one can check that $(\bigcup \mathcal{H})^*$ is a ND set. Small sets with respect to the completly Ramsey property, i.e. NR sets, can be CD, and conversely, small sets with respect to the completly doughnut property, i.e. ND sets, can be CR. So, we have to modify a proof of the base matrix lemma - see B. Balcar and P. Simon [3].

$*$-doughnuts.

For $A \subseteq^* B \subseteq \omega$ with $B \setminus \ A \in [\omega]^\omega$, we call the set

$$< A, B >^* = \{X : A \subseteq^* X \subseteq^* B\}$$

$*-$*doughnut.*

Lemma 1. *If* $\{< A_n, B_n >: n \in \omega\}$ *is a sequence of doughnuts decreasing with respect to the inclusion, then there exists a doughnut* $< C, D >$ *such that* $< C, D > \subseteq < A_n, B_n >^*$*, for each* $n \in \omega$*.*

Proof. Because of, gaps of type (ω, ω^*) and $\omega-$limits do not exist. □

$*$-disjoint.

If $< A, B >$ and $< C, D >$ are doughnuts, then the intersection $< A, B >^* \cap < C, D >^*$ is countable or has the cardinality continuum, i.e. is a

$*-$*doughnut.* Whenever $< A, B >^* \cap < C, D >^*$ is countable, then $< A, B >^*$ and $< C, D >^*$ are called $*-$*disjoint.*

Lemma 2. *If $S \in ND$, then for any doughnut $< A, B >$ there exist a doughnut $< C, D > \subseteq < A, B >$ such that $< C, D >^* \cap S^* = \emptyset$.*

Proof. The family of ND is a $\sigma-$ideal which is invariant under finite translations. □

D−partition.

Call a family $\mathcal{P}$ of $*-$doughnuts to be a $D-$*partition*, if $\mathcal{P}$ consists of $*-$doughnuts such that any two elements of $\mathcal{P}$ are $*-$disjoint, and $\mathcal{P}$ is maximal with respect to the inclusion. A $D-$partition $\mathcal{P}$ *refines* a $D-$partition $\mathcal{Q}$ (briefly $\mathcal{P} \prec \mathcal{Q}$), if for each $< A, B >^* \in \mathcal{P}$ there exists $< C, D >^* \in \mathcal{Q}$ such that $< A, B >^* \subseteq < C, D >^*$.

Shattering matrix.

Any collection of $D-$partitions is called *matrix.* A matix $\mathcal{H}$ is called *shattering*, if for each $*-$doughnut $< A, B >^*$ there exists $\mathcal{P} \in \mathcal{H}$ and $< A_1, B_1 >^*, < A_2, B_2 >^* \in \mathcal{P}$ such that $< A_1, B_1 >^* \cap < A, B >^*$ and $< A_2, B_2 >^* \cap < A, B >^*$ are different $*-$doughnuts. Let $\kappa(ND)$ be the least cardinality of a shattering matrix.

Lemma 3. *If a matix $\mathcal{H}$ is of the cardinality less than $\kappa(ND)$, then there exists a $D-$partition $\mathcal{P}$ which refines any $\mathcal{Q} \in \mathcal{H}$.*

Proof. Take a doughnut $< A, B >$. Let $\mathcal{H}(A, B)$ be a relative matrix with respect to $< A, B >$ and $\mathcal{H}$. $\mathcal{H}(A, B)$ is not shaterring in $< A, B >$. So, there exists a doughnut $< C, D > \subseteq < A, B >$ such that there exists $< A_P, B_P >^* \in \mathcal{P}$ with $< C, D >^* \subseteq < A_P, B_P >^*$, for every $\mathcal{P} \in \mathcal{H}$. Choose a $D-$partition $\mathcal{P}$ consisting of a such $< C, D >^*$. □

Corollary 4. *$\kappa(ND)$ is a regular and uncoutable cardinal number.*

Proof. One can take a shaterring matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(ND)\}$ such that $\alpha < \beta$ implies $\mathcal{P}_\beta \prec \mathcal{P}_\alpha$. Any cofinal family of $D-$ partitions from $\mathcal{H}$ constitutes a shaterring matrix, hence $\kappa(ND)$ has to be regular. $\kappa(ND)$ is uncoutable, by Lemma 1. □

D-base matrix tree.

Theorem 5. *There exists a matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(ND)\}$ which is well ordered by the inverse of $\prec$. Moreover, for each $*-$doughnut $< A, B >^*$ there is $< C, D >^* \in \bigcup \mathcal{H}$ such that $< C, D >^* \subseteq < A, B >^*$.*

Proof. Take a shaterring matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(ND)\}$ such that $\alpha < \beta$ implies $\mathcal{P}_\beta \prec \mathcal{P}_\alpha$. Let $J^c(\mathcal{P}_\alpha)$ denotes the family of all $*-$doughnuts $< A, B >^*$ such that each $< A, B >^* \in J^c(\mathcal{P}_\alpha)$ is not $*-$disjoint with continuum many elements of $\mathcal{P}_\alpha$. Let $F : J^c(\mathcal{P}_\alpha) \to \mathcal{P}_\alpha$ be an one-to-one function such that $F(G) \cap G$ is a $*-$doughnut, for every $G \in J^c(\mathcal{P}_\alpha)$. One can define F by a transfinite induction. Take a $D-$partinion

$$\mathcal{Q} \supseteq \{F(G) \cap G : G \in J^c(\mathcal{P}_\alpha)\}.$$

Having these, one can improve $\mathcal{H}$ to obtain $\mathcal{P}_{\alpha+1} \prec \mathcal{Q}$ and $\mathcal{P}_{\alpha+1} \prec \mathcal{P}_\alpha$. One obtains that, if $< A, B >^* \in J^c(\mathcal{P}_\alpha)$, then there is $< C, D >^* \in \mathcal{P}_{\alpha+1}$ with $< C, D >^* \subseteq < A, B >^*$.

We should prove that, for each $*-$doughnut $< A, B >^*$ there exist $\alpha < \kappa(ND)$ and $< C, D >^* \in \mathcal{P}_\alpha$ such that $< C, D >^* \subseteq < A, B >^*$, i.e. for each $*-$doughnut $< A, B >^*$ there exists $\alpha < \kappa(ND)$ such that $< A, B >^* \in J^c(\mathcal{P}_\alpha)$. Indeed, fix a $*-$doughnut $< A, B >^*$. Let $B^0_{\alpha_0}$ and $B^1_{\alpha_0}$ be two different $*-$doughnuts belonging to $\mathcal{P}_{\alpha_0}$ such that $D^0_{\alpha_0} =< A, B >^* \cap B^0_{\alpha_0}$ and $D^1_{\alpha_0} =< A, B >^* \cap B^1_{\alpha_0}$ are $*-$doughnuts. Thus, $D^{i_0}_{\alpha_0} \subseteq < A, B >^*$ for $i_0 \in \{0, 1\}$. Inductively, let $B^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n}$ and $B^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n}$ be two different $*-$doughnuts belonging to $\mathcal{P}_{\alpha_n}$ such that $D^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n} =< A, B >^* \cap B^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n}$ and $D^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n} =< A, B >^* \cap B^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n}$ are $*-$doughnuts. We get

$$D^{i_0 i_1 \ldots i_n}_{\alpha_n} \subseteq D^{i_0 i_1 \ldots i_{n-1}}_{\alpha_{n-1}} \subseteq < A, B >^* .$$

If $\beta = \sup\{\alpha_n : n \in \omega\}$, then $< A, B >^* \in J^c(\mathcal{P}_{\beta+1})$, by Lemma 1. $\square$

Applications to ND sets

Lemma 6. *If $\mathcal{P}$ is a $D-$partition, then the complement of the union $\bigcup\mathcal{P}$ is a ND set.*

Proof. Take a doughnut $< A, B >$. There exists $< C, D > \in \mathcal{P}$ such that $< A \cup C, B \cap D >$ is a doughnut contained in $\bigcup\mathcal{P}$. $\square$

Lemma 7. *If $S \subseteq [\omega]^\omega$ is a ND set, then there exists a $D-$partition $\mathcal{P}$ such that $\bigcup\mathcal{P} \cap S = \emptyset$.*

Proof. If S is a ND set, then S^* is a ND set, too. Thus for any doughnut $< A, B >$ there exists a doughnut $< C, D > \subseteq < A, B >$ such that $< C, D >^* \cap S^* = \emptyset$. Choose a $D-$partition $\mathcal{P}$ consisting of a such $< C, D >^*$. $\square$

Corollary 8. *$\kappa(ND) \leq add(ND)$.*

On Halbeisen's question.

In Halbeisen [8] it was stated the questions:

$$add(CD) = cov(ND)?$$

We answer, consistently under $\kappa(ND) = \mathrm{cf}\, 2^\omega$ or there are no $\kappa(ND)-$limits, this question yes. This is a counterpart to Plewik's result $add(CR) = cov(NR)$, [10].

Theorem 9. *add (CD) = cov (ND).*

Some inequalities with $add(CD)$ one can find in J. Brendle [4].

Topologization following Aniszczyk and Schilling.

Methods of B. Aniszczyk [1] or K. Schilling [11] could be adopted for doughnuts.

Theorem 10. *There exists a topology such that CD sets are exactly sets with the Baire property and ND sets are exactly nowhere dense sets.*

Therefore $add(ND) = add(CD)$. Halbeisen [8] has the same, but for pseudo-topologies.

References

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