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$\mathbf{S}_1(\Gamma, \Gamma)$ and wQN

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f_n, f – real valued functions from X into $\langle 0,1\rangle$.

$\{f_n\}_{n=0}^\infty$ **quasinormally converges** to f on a set X, written $f_n \xrightarrow{\mathrm{QN}} f$, if there exists a sequence of positive reals $\varepsilon_n \to 0$ (a **control**) such that

$$(\forall x \in X)(\exists k)(\forall n > k)\, |f_n(x) - f(x)| \le \varepsilon_n.$$

$\{f_n\}_{n=0}^\infty$ **discretely converges** to f on a set X, written $f_n \xrightarrow{\mathrm{D}} f$, if

$$(\forall x \in X)(\exists k)(\forall n > k)\, f_n(x) = f(x).$$

A topological space X is a QN-**space** if every sequence of continuous functions $f_n \to 0$ on X also $f_n \xrightarrow{\mathrm{QN}} 0$ quasi-normally on X.
A topological space X is a wQN-**space** if from every sequence of continuous functions $f_n \to 0$ on X one can choose a subsequence $f_{n_m} \xrightarrow{\mathrm{QN}} 0$ on X.
Every topological space is supposed to be perfectly normal with a countable base, if you want, even metric separable.
We can tacitly assume that X is infinite since almost all considered properties are trivial for finite spaces. $\mathcal{U} \subseteq \mathcal{P}(X)$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.
An infinite cover $\mathcal{U}$ is a γ**-cover** if every $x \in X$ lies in all but finitely many members of $\mathcal{U}$.
A γ-cover $\mathcal{U}$ is **shrinkable**, if there exists a closed γ-cover $\mathcal{V}$ that is a refinement of $\mathcal{U}$.
$\mathrm{O}(X)$, $\Gamma(X)$, $\Gamma^{co}(X)$, and $\Gamma^{sh}(X)$ – the set of all open covers, open γ-covers, clopen γ-covers, and open shrinkable γ-covers of X, respectively.
We shall deal with countable covers.
$\mathcal{A}(X)$, $\mathcal{B}(X)$ families of covers of a topological space X. X is an $\mathrm{S}_1(\mathcal{A},\mathcal{B})$**-space** if for every sequence $\{\mathcal{U}_n\}_{n=0}^\infty$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n; n \in \omega\} \in \mathcal{B}$.

Theorem 1 (M. Scheepers [Sc3])**.**
$\mathrm{S}_1(\Gamma,\Gamma)$-space is a wQN-space.

Conjecture 1. *Perfectly normal wQN-space has property $S_1(\Gamma,\Gamma)$.*

X has **SSP** if for any sequence of sequences $f_{n,m}$, $n,m \in \omega$ of continuous functions such that $f_{n,m} \to 0$ on X for every n, there exists an increasing sequence $\{m_n\}_{n=0}^\infty$ such that $f_{n,m_n} \to 0$ on X. By M. Scheepers and D. Fremlin

Theorem 2 ([Sc3] – [Fr2])**.**

$$\mathrm{wQN} \equiv \mathrm{SSP}.$$

M. Scheepers [Sc2] QN $\to (\alpha_1)$.
M. Sakai [Sa] and L. B. and J. Haleš [BH2] independently $(\alpha_1) \to$ QN, QN $\to$ $\mathrm{S}_1(\Gamma,\Gamma)$.

Theorem 3 (L.B.–J. Haleš [BH2])**.**

$$\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma^{sh},\Gamma) \equiv \mathrm{S}_1(\Gamma^{co},\Gamma).$$

$f : X \longrightarrow \mathbb{R}$ is **lower (upper) semicontinuous** if for every real a the set $\{x \in X : f(x) > a\}$ ($\{x \in X : f(x) < a\}$) is open.
X is a wQN_***-space** if from every sequence of lower semicontinuous functions $f_n \to 0$ on X one can choose a subsequence $f_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X.
wQN^***-space** … upper semicontinuous.

If $|X| < \mathfrak{b}$ then X is a wQN$_*$-space. X has **SSP$_*$** if for any sequence of sequences $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of lower semicontinuous functions such that $f_{n,m} \to 0$ on X for every n there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X.
X has **SSP*** if for any sequence of sequences $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of upper semicontinuous functions such that $f_{n,m} \to 0$ on X for every n there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X.

Theorem 4. wQN$_* \equiv$ SSP$_*$.

Proof: $f_{n,m}$ lower semicontinuous, $f_{n,m} \to 0$ on X for every n. Set

$$g_m(x) = \sum_{n=0}^{\infty} \min\{2^{-n}, f_{n,m}(x)\}.$$

g_m is lower semicontinuous and $g_m \to 0$ on X. Therefore there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $g_{m_n} \xrightarrow{\text{QN}} 0$ on X with the control $\{2^{-n}\}_{n=0}^{\infty}$. If $g_{m_n}(x) < 2^{-n}$ then

$$\min\{2^{-n}, f_{n,m_n}(x)\} = f_{n,m_n}(x) < 2^{-n}.$$

Hence $f_{n,m_n} \xrightarrow{\text{QN}} 0$ on X. If $f_m \to 0$ on X are lower semicontinuous then $2^n f_m(x) \to 0$ for every fixed n. By SSP$_*$ there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such

$$2^n f_{m_n} \to 0 \text{ on } X.$$

Then $f_{m_n} \xrightarrow{\text{QN}} 0$ on X with control $\{2^{-n}\}_{n=0}^{\infty}$.

q.e.d.

Theorem 5. SSP$^* \to$ wQN*.

Problem 1. wQN$^* \to$ SSP**?*

X has the property $\gamma\gamma_{co}$ if every γ-cover of X has a clopen γ-cover refinement (J. Haleš [Ha]). Similarly $\gamma\gamma_{sh}$.
If $\text{Ind}(X) = 0$, i.e. every two disjoint closed sets may be separated by a clopen set, then $\gamma\gamma_{co} \equiv \gamma\gamma_{sh}$.
A wQN-space with $\gamma\gamma_{sh}$ is a $\text{S}_1(\Gamma, \Gamma)$-space.

Theorem 6 (essentially J. Haleš [Ha])**.**
a) *γ-space has $\gamma\gamma_{co}$.*
b) *σ-space has $\gamma\gamma_{sh}$.*

Theorem 7. $\text{S}_1(\Gamma, \Gamma) \to$ wQN*.

Theorem 8. wQN$_* \to \text{S}_1(\Gamma, \Gamma)$.

Proof: Since wQN$_* \to$ wQN the space X has $\text{Ind}(X) = 0$. Also X is a $\text{S}_1(\Gamma^{sh}, \Gamma)$-space. It suffices to show that X has the property $\gamma\gamma_{co}$ We show

$$\text{SSP}_* \to \gamma\gamma_{co}.$$

Let $\{U_n : n \in \omega\}$ be a γ-cover. Since X is perfectly normal and $\text{Ind}(X) = 0$, there are clopen sets $F_{n,m}$ such that

$$U = \bigcup_m F_{n,m}, \qquad F_{n,m} \subseteq F_{n,m+1}.$$

We set

$$f_{n,m}(x) = \begin{cases} 1 & \text{for } x \in F_{n,m}, \\ 0 & \text{for } x \in X \setminus F_{n,m}. \end{cases}$$

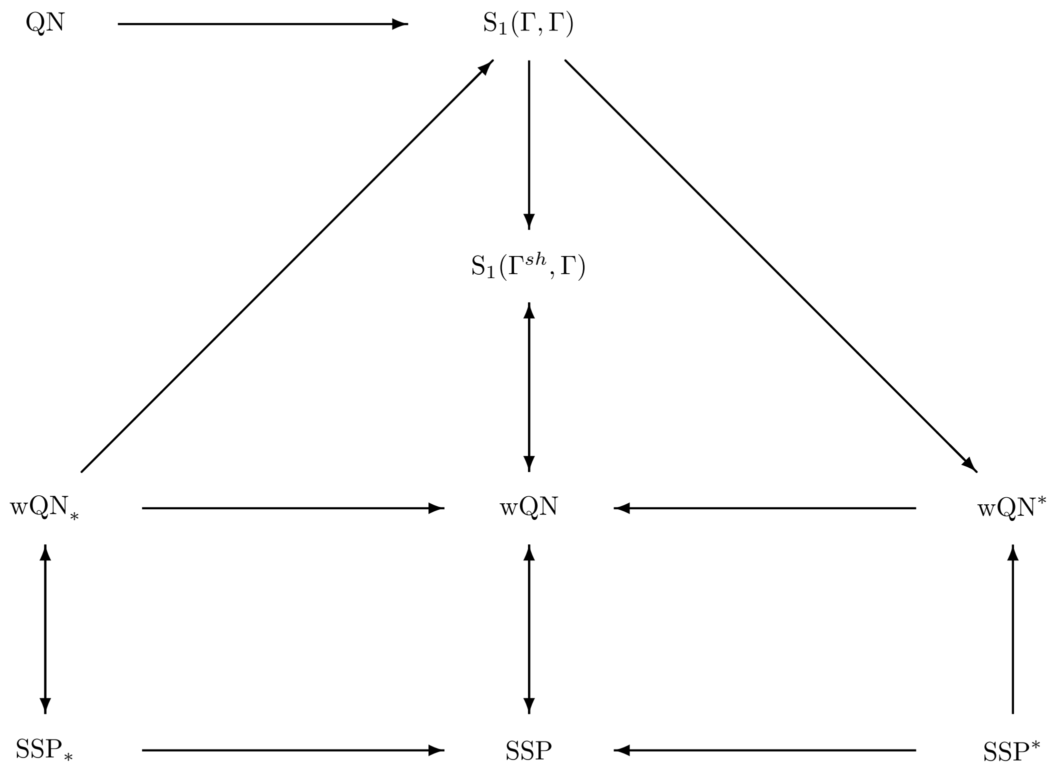
$$f_n(x) = \begin{cases} 1 & \text{for } x \in U_n, \\ 0 & \text{for } x \in X \setminus U_n. \end{cases}$$

$f_{n,m}$ is continuous, f_n is lower semicontinuous, $f_{n,m} \nearrow f_n$ and $f_n \nearrow 1$. For every fixed n we have $(f_n - f_{n,m}) \to 0$ on X. By SSP_* there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $(f_n - f_{n,m_n}) \to 0$ on X and therefore $f_{n,m_n} \to 1$. If $x \in X$ then

$$(\exists n_0)(\forall n \geq n_0)\, f_{n,m_n}(x) > \frac{1}{2},$$

i.e. for all $n \geq n_0$ we have $x \in F_{n,m_n}$.
Therefore $F_{n,m_n}, n \in \omega$ is a clopen γ-cover. Since $F_{n,m_n} \subseteq U_n$ we are done.

q.e.d.



Added after Workshop:
The results by B. Tsaban and L. Zdomskyj presented at Workshop imply that a QN-space is a wQN$_*$-space. Moreover we have realized that the implication "wQN$_*$-space $\rightarrow$ QN-space is almost trivial. Thus

$$\text{wQN}_*\text{-space} = \text{QN-space}.$$

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