Special subsets of reals characterizing local properties of function spaces

Masami Sakai

Dept. of Mathematics, Kanagawa University, Japan

1. Introduction and preliminaries.

Spaces=Tychonoff.

$\mathbb{N}, \mathbb{Q}, \mathbb{P}$ and $\mathbb{R}$ will be used to denote the positive integers, the rationals, the irrationals and the reals, respectively.

For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence.

Let $\mathfrak{c}$ be the cardinality of $\mathbb{R}$.

$\mathfrak{b}$ =min$\{|\mathcal{B}|$: $\mathcal{B}$ is an unbounded family in $\omega^\omega\}$.

$\mathfrak{p}$ = min$\{|\mathcal{F}|$: $\mathcal{F}$ is a subfamily of $[\omega]^\omega$ with the *sfip* which has no infinite pseudo-intersection$\}$.

2. The Fréchet property of $C_p(X)$

Definition. (1) X is **strictly Fréchet** if $A_n \subset X$ and $x \in \overline{A}_n$ $(n \in \omega)$ imply that there exists $\{x_n\}_{n\in\omega}$ such that $x_n \in A_n$ and $x_n \to x$.

(2) X is **Fréchet** if $A \subset X$ and $x \in \overline{A}$ imply that there exists $\{x_n\}_{n\in\omega} \subset A$ such that $x_n \to x$.

(3) X is **sequential** if for every non-closed set A in X, there exist a point $x \in X - A$ and a sequence $\{x_n\}_{n\in\omega} \subset A$ such that $x_n \to x$.

(4) X has **countable tightness** if $A \subset X$ and $x \in \overline{A}$ imply that there exists $B \subset A$ such that $x \in \overline{B}$ and $|B| = \omega$.

first countable $\Rightarrow$ strictly Fréchet $\Rightarrow$ Fréchet $\Rightarrow$ sequential $\Rightarrow$ countable tightness

Fact. $C_p(X)$ is first countable iff X is countable.

Theorem(Arhangel'skii,1976; Pytkeev, 1982). The following are equivalent.

(1) $C_p(X)$ has countable tightness;

(2) Every finite power of X is Lindelöf.

Definition. A family $\mathcal{A}$ of subsets of a set X is an ω-**cover** of X if every finite subset of X is contained in some member of $\mathcal{A}$.

Definition(Gerlits, Nagy, 1982). A space X satisfies **property (**ε**)** if every open ω-cover of X contains a countable ω-subcover of X.

Proposition(Gerlits, Nagy, 1982). Every finite power of X is Lindelöf iff X satisfies (ε).

For a sequence $\{A_n\}_{n\in\omega}$ of subsets of a set X, we put

$$\underline{Lim}\, A_n = \bigcup_{n\in\omega} \bigcap_{m\geq n} A_m.$$

Definition(Gerlits, Nagy, 1982). A space X satisfies **property (γ)** if for every open ω-cover $\mathcal{U}$ of X, there exists $\{U_n\}_{n\in\omega} \subset \mathcal{U}$ with $X = \underline{Lim}\, U_n$.

Theorem(Gerlits, Nagy, 1982; Gerlits, 1983). The following are equivalent.

(1) $C_p(X)$ is strictly Fréchet;

(2) $C_p(X)$ is Fréchet;

(3) $C_p(X)$ is sequential;

(4) X satisfies (γ).

Fact. If a space X satisfies (γ), then for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open ω-covers of X, there exists $\{U_n\}_{n\in\omega}$ such that $U_n \in \mathcal{U}_n$ and $X = \underline{Lim}\, U_n$.

We recall properties (δ) and (*) which are weaker than property (γ). If $\mathcal{A}$ is a family of subsets of a set X, then we denote by $L(\mathcal{A})$ the smallest family of subsets of X which contains $\mathcal{A}$ and is closed under the operation $\underline{Lim}$. For a sequence $\phi = \{\mathcal{U}_n\}_{n\in\omega}$ of open covers of a space X, a set $A \subset X$ is said to be ϕ-**small** if for every $n \in \omega$ there exist $k \in \omega$ and members $U_i \in \mathcal{U}_{n+i}$ $(i < k)$ with $A \subset \bigcup_{i<k} U_i$.

Definition(Gerlits, Nagy, 1982). (1) A space X has **property** (δ) if for every open ω-cover $\mathcal{U}$ of X, $X \in L(\mathcal{U})$ holds;

(2) A space X has **property** (*) if for every sequence $\phi = \{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X, X is the union of countably many ϕ-small sets.

Definition. A space X has the **Rothberger property** if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X, there exist $U_n \in \mathcal{U}_n$ $(n \in \omega)$ with $X = \bigcup_{n\in\omega} U_n$.

Theorem(Gerlits, Nagy, 1982). The following implications hold:

$(\gamma)\Rightarrow$ (δ) $\Rightarrow$ (*) $\Rightarrow$ the Rothberger property.

Every subset of $\mathbb{R}$ satisfying the Rothberger property has strong measure zero. R. Laver constructed a model of ZFC in which every set having strong measure zero is countable. Hence, in the Laver's model, every subset of $\mathbb{R}$ satisfying (γ) is countable.

Corollary(Galvin, Miller, 1984). The following equality holds:

$\mathfrak{p}$ =min$\{|X|$: $X \subset \mathbb{R}$ and $C_p(X)$ is not Fréchet$\}$.

Therefore every subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$ satisfies (γ) iff $\mathfrak{p} = \mathfrak{c}$.

[MA] If $X \subset \mathbb{R}$ and $|X| < \mathfrak{c}$, then X has property (γ).

Example(Galvin ,Miller,1984). [MA] there exists a subset $X \subset \mathbb{R}$ satisfying (γ) of cardinality $\mathfrak{c}$.

3. The κ-Fréchet property of $C_p(X)$.

Definition(Arhangel'skii). A space X is **κ-Fréchet** if for every open subset U of X and every point $x \in \overline{U}$, there exists a sequence $\{x_n\}_{n\in\omega} \subset U$ such that $x_n \to x$.

Example. The arbitrary power of $\mathbb{R}$ is κ-Fréchet.

Definition. A family $\{A_\alpha\}_{\alpha\in\eta}$ of subsets of a space X is **strongly point-finite** if for every $\alpha \in \eta$, there exists an open set U_α of X such that $A_\alpha \subset U_\alpha$ and $\{U_\alpha\}_{\alpha\in\eta}$ is point-finite.

Definition. A space X has **property** (κ) if every pairwise disjoint sequence of finite subsets of X has a strongly point-finite subsequence.

Theorem(Sakai). The following are equivalent.

(1) $C_p(X)$ is κ-Fréchet;

(2) the sequential closure of every open set of $C_p(X)$ is closed;

(3) X satisfies (κ).

Proposition. (1) Property (κ) is hereditary with respect to subspaces and finite powers;

(2) Let $f : X \to Y$ be a one-to-one continuous map. If Y satisfies (κ), then X also satisfies (κ);

(3) If every point of X has a neighborhood satisfying (κ), then X also satisfies (κ). In particular, property (κ) is preserved by the topological sum.

Proposition. Every scattered space satisfies (κ).

Definition. (1) A subset X of $\mathbb{R}$ is a λ-**set** if every countable subset of X is a G_δ-set of X.

(2) A subset X of $\mathbb{R}$ is **always of the first category** (or **perfectly meager**) for every perfect set P of $\mathbb{R}$ (i.e. P is dense in itself and closed in $\mathbb{R}$), the set $P \cap X$ is of the first category in P.

(3) A subset of $\mathbb{R}$ is a **Sierpiński set** (**Lusin set**) if it is uncountable and the intersection with every set of Lebesgue measure zero (every set of the first category) is countable.

Every Sierpiński set is a λ-set. Every λ-set is always of the first category.

Theorem(Sakai). Every λ-set of $\mathbb{R}$ satisfies (κ), and every subset of $\mathbb{R}$ satisfying (κ) is always of the first category.

Thus a Sierpiński set satisfies (κ), but a Lusin set does not satisfy (κ). We have the following implications. The implication "(*) $\Rightarrow$ always of the first category" is due to Gerlits and Nagy.

$$(\gamma) \Rightarrow (\delta) \Rightarrow (*)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

λ-set $\Rightarrow$ (κ) $\Rightarrow$ always of the first category

Example (1) [CH] there exists a space which is always of the first category and does not satisfy (κ). Let $f : \mathbb{P} \to \mathbb{P}$ be a one-to-one continuous map such that for every Lusin set $L \subset \mathbb{P}$, $f(L)$ is always of the first category. Take a Lusin set $L \subset \mathbb{P}$ and consider $f(L)$. It is always of the first category, but it does not satisfy (κ). If it had this property, L would also have this property. This is a contradiction.

(2) There exists a space satisfying (κ) which is not a λ-space. Let $[\omega]^{<\omega} \cup X \subset 2^\omega$ be the space satisfying (γ) (hence (κ)) constructed under Martin's axiom by Galvin and Miller, where the set X has cardinality continuum. For any open $U \supset [\omega]^{<\omega}$, $X - U$ has cardinality less than $\mathfrak{c}$. Therefore $[\omega]^{<\omega}$ is not a G_δ-set of $[\omega]^{<\omega} \cup X$.

Question. Does the following equality hold?:

$\mathfrak{b}$ =min$\{|X| : X \subset \mathbb{R}$ and $C_p(X)$ is not κ-Fréchet$\}$.

Rothberger proved that the equality $\mathfrak{b}$ =min$\{|X| : X \subset \mathbb{R}$ and X is not a λ-set$\}$.

Question. Let X be a λ-set which is not a λ'-set. Such a set was given in ZFC by Rothberger. Let C be a countable set of $\mathbb{R}$ such that $X \cup C$ is not a λ-set. Then $X \cup C$ is a non-λ-set which is always of the first category. Does the set $X \cup C$ satisfy (κ)?

4. The Pytkeev property and the weak Fréchet property of $C_p(X)$.

For $x \in X$, a family $\mathcal{N}$ of subsets of X is a **π-network at** x if every neighborhood of x contains some member of $\mathcal{N}$.

Definition. (1) A space X is **subsequential** if it is homeomorphic to a subspace of a sequential space;

(2) A space X is a **Pytkeev space** if $A \subset X$ and $x \in \overline{A} - A$ imply that there exists a countable π-network at x of infinite subsets of A.

(3) A space X is **weakly Fréchet** if $A \subset X$ and $x \in \overline{A} - A$ imply that there exists a pairwise disjoint family $\{F_n\}_{n \in \omega}$ of finite subsets of A such that for every neighborhood U of x, $U \cap F_n \neq \emptyset$ for all but finitely many $n \in \omega$.

subsequential $\Rightarrow$ Pytkeev $\Rightarrow$ weakly Fréchet
$\Rightarrow$ countable tightness

Theorem(Malykhin, 1999). (1) $C_p(I)$ is not subsequential, where I is the unit interval.

(2) For a compact space X $C_p(X)$ is subsequential iff it is Fréchet.

Question. Find a characterization of subsequentiality of $C_p(X)$ in terms of X.

Definition. (1) An open ω-cover $\mathcal{U}$ of a space X is **non-trivial** if $X \notin \mathcal{U}$;

(2) An open ω-cover $\mathcal{U}$ of X is ω-**shrinkable** if there exists a closed ω-cover $\{C(U) : U \in \mathcal{U}\}$ with $C(U) \subset U$ for every $U \in \mathcal{U}$.

Definition. A space X has **property (π)** if for every ω-shrinkable non-trivial open ω-cover $\mathcal{U}$ of X, there exists $\{\mathcal{U}_n\}_{n\in\omega}$ of subfamilies of $\mathcal{U}$ such that $|\mathcal{U}_n| = \omega$ and $\{\bigcap \mathcal{U}_n\}_{n\in\omega}$ is an ω-cover of X.

Theorem(Sakai, 2003). The following are equivalent.

(1) $C_p(X)$ is a Pytkeev space;

(2) X satisfies (π).

Proposition(Sakai). Property (δ) implies property (π).

Question. Does property (π) imply property (δ) (or (γ))?

Proposition. (1) Property (π) is hereditary with respect to continuous images;

(2) Let $\{X_n\}_{n\in\omega}$ be an increasing cover of X. If each X_n satisfies (π), then X also satisfies (π);

(3) If a space X satisfies (π), then the topological sum of countably many copies of X also satisfies (π).

Gerlits and Nagy noted that every space satisfying (δ) is zero-dimensional.

Fact. Every space with (π)is zero-dimensional.

Definition(Kočinac, Scheepers). A space X has the ω-**grouping property** if for every non-trivial open ω-cover $\mathcal{U}$ of X, there exists a pairwise disjoint sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of finite subfamilies of $\mathcal{U}$ such that every finite subset of X is contained in some member of $\mathcal{U}_n$ for all but finitely many $n \in \omega$.

Definition. A space X has **property (wγ)** if for every ω-shrinkable non-trivial open ω-cover $\mathcal{U}$ of X, there exists a pairwise disjoint sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of finite subfamilies of $\mathcal{U}$ such that every finite subset of X is contained in some member of $\mathcal{U}_n$ for all but finitely many $n \in \omega$.

Theorem(Sakai, 2003). Tthe following are equivalent.

(1) $C_p(X)$ is weakly Fréchet;

(2) X satisfies ($w\gamma$).

Corollary. If X has the ω-grouping property, then $C_p(X)$ is weakly Fréchet.

Question. Does property (**w**γ) imply the ω-grouping property?

Proposition. (1) Property ($w\gamma$) is hereditary with respect to continuous images;

(2) If a space X satisfies ($w\gamma$), then the topological sum of countably many copies of X also satisfies ($w\gamma$).

A subset A of $\mathbb{R}$ is said to have **universal measure zero** if for every Borel measure μ on $\mathbb{R}$ there exists a Borel set B with $A \subset B$ and

$\mu(B) = 0$, where a Borel measure means a countably additive, atomless (i. e. $\mu(\{x\}) = 0$ for each $x \in X$), finite measure.

Theorem(Sakai, 2003). Let X be a subset of $\mathbb{R}$. If X satisfies (π), then X has universal measure zero and is always of the first category.

Theorem(Sakai). Let $\mathcal{B}$ be a non-trivial countable ω-cover of Borel sets of $\mathbb{P}$. Then there exists a pairwise disjoint sequence $\{\mathcal{B}_n\}_{n\in\omega}$ of finite subfamilies of $\mathcal{B}$ such that every finite subset of X is contained in some member of $\mathcal{B}_n$ for all but finitely many $n \in \omega$. In particular, every analytic set has the ω-grouping property.

Corollary. $C_p(\mathbb{P})$ is weakly Fréchet.

Proposition. For every free ultrafilter $\mathcal{F}$ on ω, $C_p(\mathcal{F})$ is not weakly Fréchet.

Kočinac and Scheepers conjectured that $\mathfrak{b}$ is the minimal cardinality of a set $X \subset \mathbb{R}$ such that $C_p(X)$ is not weakly Fréchet.

Theorem(Tsaban, 2004). The following equality holds:

$\mathfrak{b}$ =min$\{|X| : X \subset \mathbb{R}$ and $C_p(X)$ is not weakly Fréchet$\}$.

Question. Determine the minimal cardinality of a set $X \subset \mathbb{R}$ such that X does not satisfy (π).

5. Tightness-like properties of $C_p(X)$.

Definition. (1) (Arhangel'skii) A space X has **countable fan tightness** if $A_n \subset X$ and $x \in \overline{A}_n$ ($n \in \omega$) imply that there exists $\{F_n\}_{n\in\omega}$ of finite subsets such that $F_n \subset A_n$ and $x \in \overline{\bigcup_{n\in\omega} F_n}$.

(2) (Sakai) A space X has **countable strong fan tightness** if $A_n \subset X$ and $x \in \overline{A}_n$ ($n \in \omega$) imply that there exists $\{x_n\}_{n\in\omega}$ such that $x_n \in A_n$ and $x \in \overline{\{x_n : n \in \omega\}}$.

strictly Fréchet $\Rightarrow$ countable strong fan tightness $\Rightarrow$ countable fan tightness $\Rightarrow$ countable tightness.

Definition. A space X has the **Menger property** if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X, there exist a finite subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ ($n \in \omega$) such that $\bigcup_{n\in\omega} \mathcal{U}_n$ is a cover of X.

Theorem(Arhangel'skii, 1986). The following are equivalent.

(1) $C_p(X)$ has countable fan tightness;

(2) Every finite power of X has the Menger property.

The condition (2) in this theorem can be characterized in terms of an ω-cover.

Proposition(Just, Miller, Scheepers, Szeptycki, 1996). The following are equivalent.

(1) Every finite power of X has the Menger property;

(2) For every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open ω-covers of X, there exist a finite subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega}\mathcal{U}_n$ is an ω-cover of X.

Theorem(Sakai, 1988). The following are equivalent.

(1) $C_p(X)$ has countable strong fan tightness;

(2) Every finite power of X has the Rothberger property;

(3) For every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open ω-covers of X, there exist $U_n \in \mathcal{U}_n$ $(n \in \omega)$ such that $\{U_n\}_{n\in\omega}$ is an ω-cover of X.

Proposition(Sakai). Every Pytkeev space with countable fan tightness has countable strong fan tightness.

Corollary. Let X be a space satisfying (π). If every finite power of X has the Menger property, then every finite power of X has the Rothberger property.

6. The results of Kočinac and Scheepers.

Definition. A space X has the **Hurewicz property** if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X, there exist a finite subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Proposition(Kočinac, Scheepers, 2003). The following are equivalent.

(1) Every finite power of X has the Hurewicz property;

(2) Every finite power of X has the Menger property and X has the ω-grouping property;

(3) For every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open ω-covers of X, there exist a finite subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that every finite subset of X is contained in some member of $\mathcal{V}_n$ for all but finitely many $n \in \omega$.

The following local property of a space was considered by Kočinac and Scheepers.

Definition. A space X is **weakly Fréchet in the strict sence** if $A_n \subset X$ and $x \in \overline{A}_n$ ($n \in \omega$) imply that there exists $\{F_n\}_{n\in\omega}$ of finite subsets such that $F_n \subset A_n$ and for every neighborhood U of x, $U \cap F_n \neq \emptyset$ for all but finitely many $n \in \omega$.

Theorem(Kočinac, Scheepers, 2003). For a space X, the following are equivalent.

(1) $C_p(X)$ is weakly Fréchet in the strict sence;

(2) $C_p(X)$ has countable fan tightness and is weakly Fréchet;

(3) Every finite power of X has the Hurewicz property.

Note that $C_p(\mathbb{P})$ is weakly Fréchet, but it is not weakly Fréchet in the strict sence.

Proposition(Kočinac, Scheepers, 2003). The following equality holds:

$\mathfrak{b}$ =min$\{|X| : X \subset \mathbb{R}$ and $C_p(X)$ is not weakly Fréchet in the strict sence $\}$.

Let us recall property (*) considered in the second section. Nowik, Scheepers and Weiss gave a characterization of property (*).

Theorem(Nowik, Scheepers, Weiss, 1998). The following are equivalent.

(1) X has property (*);

(2) X has both the Hurewicz property and the Rothberger property.

Hence Kočinac and Scheepers proved:

Theorem(Kočinac, Scheepers, 2002). The following are equivalent.

(1) $C_p(X)$ has countable strong fan tightness and is weakly Fréchet;

(2) Every finite power of X has property (*).

7. AP and WAP properties of $C_p(X)$.

An AP-space was introduced Pultr and Tozzi(1993) and a WAP-space was considered by Simon(1994). "AP" is "Approximation by Points" and "WAP" is "Weak Approximation by Points".

Definition. (1) A space X is an **AP-space** if for every $A \subset X$ and every point $x \in \overline{A} - A$, there exists a set $B \subset A$ such that $\overline{B} = B \cup \{x\}$.

(2) A space X is a **WAP-space** if for every non-closed set A in X, there exist a point $x \in \overline{A} - A$ and a set $B \subset A$ such that $\overline{B} = B \cup \{x\}$.

Obviously we have the following implications:

$$\begin{array}{ccc} \text{Fréchet} & \Rightarrow & \text{sequential} \\ \Downarrow & & \Downarrow \\ \text{AP} & \Rightarrow & \text{WAP} \end{array}$$

Bella and Yaschenko gave a space X (1999) such that $C_p(X)$ is WAP, but not AP.

A space X is ω-ψ-**monolithic** if the closure of every countable subset of X has countable pseudocharacter in itself.

Lemma(Bella, Yaschenko, 1999). Every ω-ψ-monolithic space with countable fan tightness is an AP-space.

Theorem(Bella, Yaschenko, 1999). If a space X is σ-compact, then $C_p(X)$ is an AP-space.

If X is separable, then $C_p(X)$ has countable pseudocharacter. Hence, if $X \subset \mathbb{R}$ and every finite power of X has the Menger property, then $C_p(X)$ is an AP-space.

On the other hand:

Theorem(Tkachuk, Yaschenko, 2001). If $C_p(X)$ is an AP-space and X is paracompact, then X has the Menger property.

Corollary. $C_p(\mathbb{P})$ is not an AP-space.

There exist many open questions on the AP-property and the WAP-property of $C_p(X)$.

We should note that the WAP-property is important when we study the M_3-M_1-problem. It is a famous longstanding open problem in general topology which asks whether every M_3-space is M_1. Let $C_k(X)$ be the space of all continuous real-valued functions on a space X with the compact-open topology. Gartside and Reznichenko proved in 2000 that the space $C_k(\mathbb{P})$ is an M_3-space. So far, it is open whether $C_k(\mathbb{P})$ is M_1. Mizokami et al. showed in 2001 that every M_3-space with the WAP-property is an M_1-space. Therefore it is important to investigate the WAP-property of $C_k(\mathbb{P})$. It is open whether $C_k(\mathbb{P})$ is a WAP-space.

8. Arhangel'skii's α_i-properties of $C_p(X)$.

Definition(Arhangel'skii, 1972). For i =1, 2, 3 and 4, a space X is an α_i-**space** if for every countable family $\{S_n\}_{n\in\omega}$ of sequences converging to some point $x \in X$, there exists a sequence S converging to x such that:

(α_1) $S_n - S$ is finite for all $n \in \omega$;

(α_2) $S_n \cap S$ is infinite for all $n \in \omega$;

(α_3) $S_n \cap S$ is infinite for infinitely many $n \in \omega$;

(α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

Theorem(Scheepers, 1998). $C_p(X)$ has property α_2 iff it has property α_4.

An open cover $\mathcal{U}$ of a space X is a γ-**cover** if every point of X is contained in all but finitely many members of $\mathcal{U}$.

Theorem(Scheepers, 1999). For a perfectly normal space X, the following are equivalent.

(1) $C_p(X)$ is an α_2-space;

(2) For every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of γ-covers of X, there exist $U_n \in \mathcal{U}_n$ $(n \in \omega)$ such that $\{U_n\}_{n\in\omega}$ is a γ-cover of X.

Definition. (1) (Bukovská, 1991). Let f and $f_n(n \in \omega)$ be real-valued functions on a space X. We say that $\{f_n\}_{n\in\omega}$ **converges quasinormally** to f if there exists a sequence $\{\epsilon_n\}_{n\in\omega}$ of positive real numbers such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ and for each $x \in X$ $|f_n(x) - f(x)| < \varepsilon_n$ holds for all but finitely many $n \in \omega$.

(2) (Bukovský, Reclaw, Repiský, 1991). A space X is a **QN-space** if whenever a sequence $\{f_n\}_{n\in\omega} \subset C_p(X)$ converges to $f \in C_p(X)$, the convergence is quasinormal convergence.

Theorem(Scheepers, 1998). If $C_p(X)$ is an α_1-space, then X is a QN-space.

Reclaw proved that every QN-space of real numbers is a σ-set. On the other hand, Scheepers showed that for a Sierpiński set X, $C_p(X)$ is an α_1-space.

Question. Find a characterization of α_1-space $C_p(X)$ in terms of X.

Proposition(Scheepers, 1998). The following equality holds:

$\mathfrak{b}$ =min$\{|X| : X \subset \mathbb{R}$ and $C_p(X)$ is not an α_1-space$\}$.