Families of Trigonometric Thin Sets and Related Exceptional Sets

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Trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx), \qquad (1)$$

$a_n,\ b_n,\ n \in \omega$ reals, $b_0 = 0$.

A. Denjoy and N. N. Luzin, Comptes Rendus des l'Académie des Sciences de Paris, 1912, identical title

Sur l'absolue convergence
des séries trigonométriques

Theorem 1 (A. Denjoy − N. N. Luzin) *If series* (1) *absolutely converges on a set of positive Lebesgue measure, then*

$$\sum_{n=0}^{\infty} (|a_n| + |b_n|) < \infty,$$

i.e. the series (1) *absolutely converges everywhere.*

Theorem 2 (N. N. Luzin) *If series* (1) *absolutely converges on a non-meager set, then*

$$\sum_{n=0}^{\infty} (|a_n| + |b_n|) < \infty,$$

i.e. the series (1) *absolutely converges everywhere.*

A set $A \subseteq \mathbb{T}$ is an **AC**-set if every trigonometric series (1) converging absolutely on A converges absolutely everywhere.

An **N**-set = not an **AC**-set.

R. Salem: A is an **N**-set if and only if there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative reals such that $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n \|nx\| < \infty$ for every $x \in A$

G is a locally compact topological group, $\widehat{G}$ its dual group. Elements of $\widehat{G}$ are **characters**. Banach space $\mathrm{C}^*(X)$ of continuous bounded real functions on $X \subseteq G$ with the norm

$$\|f\| = \sup\{|f(x)|; x \in X\}.$$

A set X is a **Dirichlet set** if 1_X belongs to the closure of $\widehat{G}|X$.

X is a Dirichlet set if and only if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\|n_k\|$ converges uniformly to 0 on X.

$\mathrm{C}^*(X)^*$ is the dual space of $\mathrm{C}^*(X)$. Weak topology: the weakest topology in which every $F \in \mathrm{C}^*(X)^*$ is continuous. A locally compact set $X \subseteq G$ is a weak Dirichlet set if 1_X belongs to the closure of $\widehat{G}|X$ in the weak topology.

Riesz Theorem: $\mathsf{C}^*(X)^*$ is the space of all Borel measures on X. X is a weak Dirichlet set if and only for every positive Borel measure μ on X there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_X \|n_k x\| \, d\mu(x) = 0.$$

Generally a set $X \subseteq \mathbb{T}$ is a **weak Dirichlet set** (shortly **wD-set**) if there exists a universally measurable set $X \subseteq \mathbb{T}$ such that for every positive Borel measure μ on B there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_B \|n_k x\| \, d\mu(x) = 0.$$

The families will be denoted $\mathcal{D}$, $\mathcal{N}$, and $w\mathcal{D}$.

Dirichlet set $\to$ N-set $\to$ weak Dirichlet set.

A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$ is a **family of thin sets** iff
(a) $\{x\} \in \mathcal{F}$ for every $x \in \mathbb{T}$;
(b) if $B \subseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$;
(c) no interval belongs to $\mathcal{F}$.

Theorem 3 *$\mathcal{N}$ is a family of thin sets.*

A set $\mathcal{G} \subset \mathcal{F}$ is a **basis of** $\mathcal{F}$ if for every $A \in \mathcal{F}$ there exists a set $B \in \mathcal{G}$ such that $A \subseteq B$.
$\{A \in \mathcal{N}; A$ is a F_σ set$\}$ is a basis of $\mathcal{N}$.
Arithmetical difference

$$A - A = \{z \in \mathbb{T}; (\exists x, y \in A)\, z = x - y\}.$$

Theorem 4 (H. Steinhaus) *If A has positive measure or possesses the Baire property and in not meager then $A - A$ contains an open interval.*

Theorem 5 *If a family of thin sets $\mathcal{F}$ with a Borel basis is closed under arithmetical difference then every set from $\mathcal{F}$ is meager and has Lebesgue measure zero.*

$A \subseteq \mathbb{T}$ is **permitted** for $\mathcal{F}$ if for any $B \in \mathcal{F}$ also $A \cup B \in \mathcal{F}$. Perm($\mathcal{F}$) denotes the family of all permitted sets for $\mathcal{F}$.
Perm($\mathcal{F}$) $\subseteq \mathcal{F}$ is an ideal;
Perm($\mathcal{F}$) $= \mathcal{F}$ if and only if $\mathcal{F}$ is an ideal.

Theorem 6 (J. Arbault – P. Erdös) *Every countable subset of $\mathbb{T}$ is permitted for $\mathcal{N}$.*

A set $A \subseteq \mathbb{T}$ is called:
a **pseudo Dirichlet set** (shortly **pD**-set), if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges *quasinormally* to 0 on A;
an **A**-set ... *pointwise* ...;
an $\mathbf{N}_0$-set if there is an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\sum_{k=0}^{\infty} \|n_k x\| < \infty$ on A;
an $\mathbf{B}_0$-set if there is a real $c > 0$ and ... $\sum_{k=0}^{\infty} \|n_k x\| < c$ on A;
a **B**-set if there is a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative reals, $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n \|nx\| < c$.
The corresponding families: $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{B}$.
In all the definitions the function $\|x\|$ can be replaced by $|\sin \pi x|$.

Theorem 7

(i)*Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ is a family of thin sets and the following inclusions hold true:*

$$\begin{array}{ccccc} & & \mathcal{A} & \longrightarrow & w\mathcal{D} \\ & & \uparrow & & \uparrow \\ p\mathcal{D} & \longrightarrow & \mathcal{N}_0 & \longrightarrow & \mathcal{N} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D} & \longrightarrow & \mathcal{B}_0 & \longrightarrow & \mathcal{B} \end{array}$$

(ii) *Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ is closed under arithmetical difference and therefore contains only meager sets of Lebesgue measure zero.*

(iii) *Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ has a Borel basis.*

(iv) *Every family $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$ contains each countable subset of $\mathbb{T}$.*

(v) *Every finite subset of $\mathbb{T}$ is permitted for $\mathcal{D}$, $\mathcal{B}_0$, $\mathcal{B}$.*

(vi) *Every countable subset of $\mathbb{T}$ is permitted for $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.*

$f, g : \mathbb{T} \longrightarrow \langle 0, 1\rangle$ continuous, $f(0) = g(0) = 0$.
Replacing the $\| \; \|$ function in the above definitions by a function f we obtain a f-**Dirichlet set** (**D**$_f$**-set**), **pseudo** f-**Dirichlet set** (**pD**$_f$**-set**), **A**$_f$**-set**, **N**$_{0\,f}$**-set**, **B**$_{0\,f}$**-set**, **N**$_f$**-set**, **B**$_f$**-set**, and **weak** f-**Dirichlet set** (**wD**$_f$**-set**). Similar inclusions as above hold true for those families and every countable set is a **pD**$_f$**-set**, i.e. the conclusions (i) and (iii) of the theorem hold true. For the conclusion (ii) one needs some additional condition.
The **zero**-set $\mathbf{Z}(f) = \{x \in \mathbb{T}; f(x) = 0\}$ of f.

Theorem 8 (Z. Bukovská) *If* $n \cdot \mathbf{Z}(f) \subseteq \mathbf{Z}(g)$ *for some positive integer* n *then* $\mathcal{F}_f \subseteq \mathcal{F}_g$ *for* $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.

Corollary 9 *If* $\mathbf{Z}(f)$ *is a finite set of rationals then* $\mathcal{F} = \mathcal{F}_f$ *for* $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.

Corollary 10 $\mathcal{D} \subseteq \mathcal{F}_f$ *for any* $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.

Theorem 11 (Z.B. – L.B.) *Assume that*

$$(\forall x,\, |x| < 1/2)\, f(x/m) \le f(x)$$

for any positive integer m *and* $\mathbf{Z}(g)$ *is a finite set of rationals. Then* $\mathcal{N}_{0f} \subseteq \mathcal{N}_{0g}$ *if and only if*

$$(\forall \{x_k\}_{k=0}^{\infty}) \left(\sum_{k=0}^{\infty} f(x_k) < \infty \to \sum_{k=0}^{\infty} g(x_k) < \infty \right).$$

Problem: What is the smallest size of a basis of a family of thin sets?
Following an idea of J. Marcinkiewicz

Lemma 12 (L.B.) *There exists a family* $\mathcal{M}$ *of Dirichlet sets,* $|\mathcal{M}| = \mathfrak{c}$ *such that* $A - B$ *contains a non-trivial interval for any* $A, B \in \mathcal{M}$, $A \neq B$.

Theorem 13 *Let* $\mathcal{F}$ *be a family of thin sets such that* $\mathcal{D} \subseteq \mathcal{F}$ *and there exists a family of thin sets* $\mathcal{H}$ *closed under arithmetic difference and such that* $\mathcal{F} \subseteq \mathcal{H}$. *Then any basis of the family* $\mathcal{F}$ *has cardinality at least* $\mathfrak{c}$.

Corollary 14 *Every basis of any trigonometric family of thin sets has cardinality at least* $\mathfrak{c}$.

L. B. – N.Kholshchevnikova – M. Repický:

Theorem 15 *Every γ-set is permitted for any of the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$.*

Theorem 16 (Dirichlet – Minkowski)
If $\{n_i\}_{i=0}^{\infty}$ is an increasing sequence of natural numbers then for any reals $x_1, \ldots, x_k \in \mathbb{T}$ and any $\varepsilon > 0$, there are $i, j \in \omega$ such that $0 \leq i < j \leq (2/\varepsilon)^k$ and

$$\|(n_j - n_i)x_l\| < \varepsilon \qquad \text{for } l = 1, 2, \ldots, k. \tag{2}$$

Actually theorem says that 0_B belongs to the closure of the set $\{\|(n_i - n_j)x\|; i \neq j\}$ in the topology of pointwise convergence.

Metatheorem 1 *It is consistent with* ZFC *that there exists a permitted set for any of the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$ of size $\mathfrak{c}$.*

J. Arbault: "proof" of the existence of a perfect permitted set for $\mathcal{N}$.
N. K. Bary: a gap in the proof.
M. Repický: a set A has **perfect measure zero** if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there is an increasing sequence of natural numbers $\{n_k\}_{k-0}^{\infty}$ and a sequence of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $|\mathcal{I}_n| \leq n$, $|I| < \varepsilon_n$ for every $I \in \mathcal{I}_n$, and $A \subseteq \bigcup_m \bigcap_{k>m} \bigcup \mathcal{I}_{n_k}$.
γ-set has perfect measure zero and set of perfect measure zero has strong measure zero.

Theorem 17 (M. Repický) *Let $\mathcal{F}$ be any of the families $\mathcal{N}$, $\mathcal{A}$, $\mathcal{N}_0$, and $p\mathcal{D}$. The unions of less than $\mathfrak{t}$ sets having perfect measure zero are permitted for $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$,and $\mathcal{A}$.*

Conjecture 18 (L.B.) *Every set permitted for $\mathcal{A}$ or $\mathcal{N}$ is perfectly meager.*

Theorem 19
(P. Erdös − K. Kunen − R. D. Mauldin)
If $P \subseteq \mathbb{T}$ is perfect set then there exists a perfect set Q of measure zero such that $P + Q = \mathbb{T}$.

Theorem 20 (P. Eliaš) *Let $P \subseteq \mathbb{T}$ be a perfect set. Then there exists a pseudo Dirichlet set Q such that the set $P \cap (x - Q)$ is dense in P for every $x \in \mathbb{T}$.*

Corollary 21 (P.E.) *Let $P \subseteq \mathbb{T}$ be a perfect set. Then there exists a pseudo Dirichlet set Q such that $P + Q = \mathbb{T}$.*

Theorem 22 (P.E.) *Let $\mathcal{F}$ be a family of thin sets with a F_σ basis containing every pseudo Dirichlet set. If $\mathcal{F}$ is closed under arithmetical difference then every $\mathcal{F}$–permitted set is perfectly meager.*

Proof: Assume that $A \subseteq \mathbb{T}$ is an $\mathcal{F}$–permitted set, $P \subseteq \mathbb{T}$ is perfect. By theorem 20 there exists a pseudo Dirichlet set Q such that $P \cap (x - Q)$ is dense in P for every $x \in \mathbb{T}$. By assumption about $\mathcal{F}$ we have $Q \in \mathcal{F}$ and therefore $A \cup Q \in \mathcal{F}$. Thus, there exists an F_σ set $B \in \mathcal{F}$, $B \supseteq A \cup Q$. Since $\mathcal{F}$ is closed under arithmetical difference we have $B - B \neq \mathbb{T}$. Then there exists an $x \in \mathbb{T}$ such that $B \cap (x - B) = \emptyset$.

Then also $B\cap(x-Q)=\emptyset$ and therefore $P\cap(x-Q)$ is a subset of G_δ set $P\setminus B$ dense in P. Hence $P\cap A$ is meager.

q.e.d.

Theorem 23 (P. Eliaš) *Every set permitted for any of the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$ is perfectly meager.*

Proof: For any of the families $p\mathcal{D}$, $\mathcal{N}_0$ and $\mathcal{N}$ the assertion follows directly from theorem 22. For A-sets we must modify the proof.

Let A, P, Q be as above. Since $A\cup Q$ is an A-set there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $A\cup Q\subseteq\{x\in\mathbb{T};\|n_kx\|\to 0\}$. Denote

$$B_i=\{x\in\mathbb{T};(\forall k\geq i)\|n_kx\|\leq 1/8\},\qquad B=\bigcup_i B_i.$$

Then B is an F_σ set and $A\cup Q\subseteq B$. If $x\in B-B$ then there are i_1,i_2 and $x_1\in B_{i_1}$, $x_2\in B_{i_2}$ such that $x=x_1-x_2$. If $i_0=\max\{i_1,i_2\}$ then $x_1,x_2\in B_{i_0}$ and therefore $x\in B_{i_0}-B_{i_0}$. Thus $B-B=\bigcup_i(B_i-B_i)$. On the other hand we have $B_i-B_i\subseteq B_{i+1}-B_{i+1}$ and

$$B_i-B_i\subseteq\{x\in\mathbb{T};\|n_ix\|\leq 1/4\}.$$

One can easily see that

$$\lambda(\{x \in \mathbb{T}; \|nx\| \leq 1/4\}) = 1/2$$

and therefore $\lambda(B_n - B_n) \leq 1/2$ for any $n > 0$. Thus $\lambda(B - B) \leq 1/2$. Hence $B - B \neq \mathbb{T}$ and we can continue as in the proof of theorem 22.

q.e.d.

Metatheorem 2 ZFC + "every set permitted for any of the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$ has cardinality $\leq \aleph_1$" is consistent.

Metatheorem 3 *"Every set of cardinality $< \mathfrak{c}$ is permitted for the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$" is undecidable in* ZFC.

Metatheorem 4 *"There exists a permitted set for any of the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$ of cardinality $\mathfrak{c}$" is undecidable in* ZFC.

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