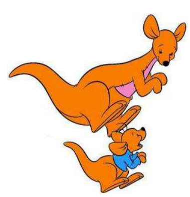
Numbers

and

Colors

A tour of Ramsey Theory



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Boaz Tsaban

Preface

This book covers the fundamental theorems concerning colorings of the natural numbers and related structures. It grew out of lecture notes for a course delivered repeatedly, on student reuqest, at the Weizmann Institute of Science and at Bar-Ilan University. I first delivered it using Protasov's elegant booklet *Combinatorics of Numbers*, and this influence is still visible. I then included more advanced theorems, adapted from Hindman and Strauss's thorough monograph *Algebra in the Stone–Čech Compactification*. Finally, I added some results that do not appear in earlier books. Historical comments are based on the above-mentioned books, and on Alexander Soifer's chapter *Ramsey theory before Ramsey, prehistory and early history: an essay in 13 parts*, in the book *Ramsey Theory: Yesterday, Today, and Tomorrow*. Neil Hindman has been of great help in detecting references for results I thought were new (oh, well).

This book is suitable both for self-study and as a textbook for a course. It can also serve as a launching point for independent research in this beautiful field.

I made substantial efforts to simplify the proofs and notations used in other accounts, and to make the material accessible to any good second year undergraduate student. The very few places where some knowledge or notion from later undergraduate years is needed can either be skipped or taken for granted. In few cases, a well-known notion is mentioned without its definition, and if needed, the corresponding Wikipedia entry is sufficient to catch up.

It is highly recommended to solve or at least consider the exercises scattered through the text. Proper understanding would be harder otherwise.

This book contains a number of *excursions*: sections that are not necessary for the remainder of the book, but highlight interesting additional aspects of the studied material. These sections may serve as refreshment breaks. Even a tulip gardener is glad to occasionally see additional kinds of flowers.

Comments at the end of chapters are not meant to be self-contained, and may be skipped by readers not familiar with the concepts mentioned there.

Most results presented here cannot be generalized in an obvious manner. Counter-examples are often available in the monograph of Hindman and Strauss, that I recommended to any reader beginning his research in this area.

I thank my students for their many useful comments and suggestions. This book is dedicated to my children.

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CHAPTER 1

Three famous coloring theorems

Ramsey theory, named after Frank P. Ramsey (1903–1930), studies the following phenomenon: If we take a rich mathematical structure, and color (no matter how) each of its elements in one out of finitely many prescribed colors, there will be a rich *monochromatic* substructure, that is, a rich substructure with all elements of the same color. In this chapter, we provide elementary proofs of several beautiful theorems exhibiting this phenomenon.

1. Ramsey's Theorem

For a set A and a natural number d, let $[A]^d := \{ F \subseteq A : |F| = d \}$, the collection of all d-element subsets of A.

A *graph* is a pair $G = (V, E)$, consisting of a set of vertices V and a set E of edges among vertices. Formally, E is a subset of $[V]^2$, and $\{a, b\} \in E$ is interpreted as "there is an edge between a and b". The graph G is *complete* if $E = [V]^2$, that is, there is an edge between every pair of vertices.

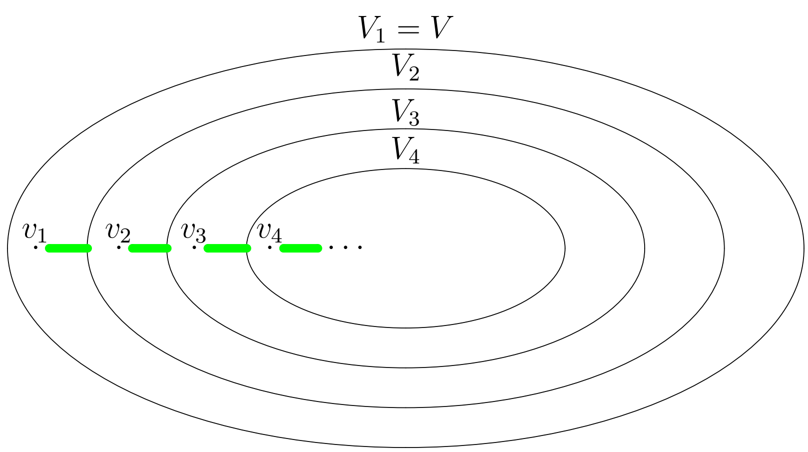
THEOREM 1.1 (Ramsey). *If we color each edge of an infinite complete graph with one out of finitely many prescribed colors, then there is an infinite complete monochromatic subgraph. That is, an infinite set of vertices with all edges among them of the same color.*

PROOF. The following proof is Ramsey's original. Several alternative proofs were suggested since, but, remarkably, Ramsey's proof remains the most lucid one.

We prove the theorem in the case of two colors, and later see how to generalize it to an arbitrary finite number of colors. Assume, thus, that the colors are red and green.

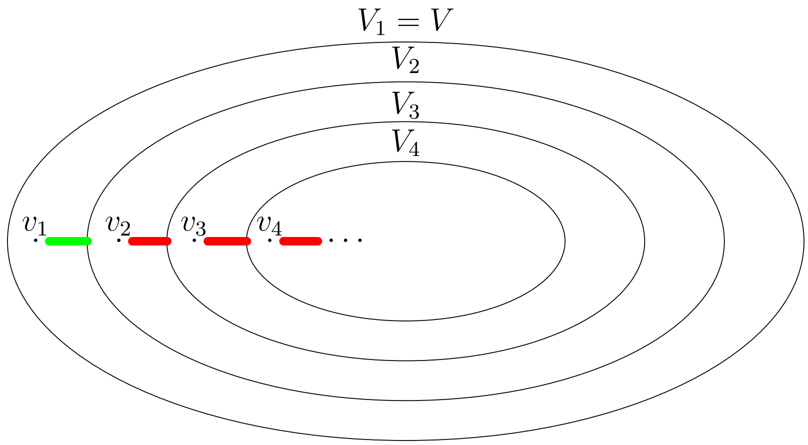
Every two vertices are joined by an edge, a red one or a green one. Let $V_1 := V$. Choose a vertex $v_1 \in V_1$. Either this vertex has infinitely many green edges, or it has infinitely many red edges. We may assume that the case is the former. Let V_2 be the (infinite) set of vertices connected to v_1 by green edges. There are two cases to consider.

The good case:



Assume that there is a vertex $v_2 \in V_2$ with infinitely many green edges connecting it to other vertices in V_2. Let $V_3 \subseteq V_2$ be the set of these vertices. Continue by induction, as long as possible: For each n, assume that there is a vertex $v_n \in V_n$ with infinitely many green edges connecting it to vertices in V_n, and let $V_{n+1} \subseteq V_n$ be the set of these vertices. If this is the case for all n, then all edges of the complete graph with vertices $v_1, v_2, \dots$ are green. Indeed, for $n < m$, we have $v_m \in V_{n+1}$, and thus the edge $\{v_n, v_m\}$ is green.

The remaining, even better case:



Assume that, for some n, the above procedure terminates: each $v \in V_n$ has only finitely many green edges. In this case, we restart the procedure, from an arbitrary vertex $v_n \in V_n$, with *red* edges. This time, the procedure cannot terminate since, for $m \geq n$, and each vertex $v_m \in V_m$, all but finitely many edges connecting v_m to other elements of V_m are red! Thus, in the remaining case, we obtain vertices $v_n, v_{n+1}, \dots$ with all edges among them red. $\square$

For brevity, we make some terminological conventions. Let A be a nonempty set. By *coloring of A* we mean a coloring of the elements of A, each by one color out of a prescribed set of colors. A *finite coloring* of A is a coloring in a finite number of colors. A *k-coloring* of A is a coloring in k colors.

Whenever convenient, we will identify each color with a natural number. For example, a coloring of a set A with colors red, green and blue is a function $c\colon A \to \{\text{red}, \text{green}, \text{blue}\}$, and we may consider instead a function $c\colon A \to \{1, 2, 3\}$.

In accordance with our earlier uses of the word, a *monochromatic* set is a set with all elements of the same color.

Exercise 1.2. We have proved Ramsey's Theorem for 2-colorings. Prove it for arbitrary finite colorings.
Hint: A color blindness argument.

In Ramsey's Theorem 1.1, we may restrict attention to a countable subgraph of the given graph, and enumerate its vertices by the natural numbers. Thus, we may assume that $V = \mathbb{N}$. The following theorem generalizes Theorem 1.1.

Theorem 1.3 (Ramsey). *Let d be a natural number. For each finite coloring of $[\mathbb{N}]^d$, there is an infinite set $A \subseteq \mathbb{N}$ such that $[A]^d$ is monochromatic.*

Proof. By induction on d. The case $d = 1$ is immediate. For $d > 1$, given a coloring

$$c\colon [\mathbb{N}]^d \to \{1, \dots, k\},$$

choose $v_1 \in \mathbb{N}$ and consider the coloring $c_{v_1}\colon [\mathbb{N} \setminus \{v_1\}]^{d-1} \to \{1, \dots, k\}$, defined by

$$c_{v_1}(\{v_2, \dots, v_d\}) = c(\{v_1, v_2, \dots, v_d\}).$$

By the inductive hypothesis, there is an infinite set $V_1 \subseteq \mathbb{N} \setminus \{v_1\}$ such that $[V_1]^{d-1}$ is monochromatic for the coloring c_{v_1}.

Continue as in the proof of Theorem 1.1. □

Exercise 1.4. Complete the proof of Theorem 1.3.

Exercise 1.5. Let A be an infinite set of points in the plane, such that each line contains at most a finite number of points from A. Using Ramsey's Theorem, prove that there is an infinite set $B \subseteq A$ such that each line contains at most two points from B.

2. Compactness, the Four Color Theorem, and the Finite Ramsey Theorem

Each of the coloring theorems we prove has a variation where the colored set is finite. These finite variations follow from the infinite ones, thanks to the following result.

Theorem 2.1 (Compactness). *Let $X = \{x_1, x_2, \dots\}$ be a countable set, and let $\mathcal{A}$ be a family of finite subsets of X. Assume that for each k-coloring of X, there is in $\mathcal{A}$ a monochromatic set. Then there is a natural number n such that for each k-coloring of the set $\{x_1, \dots, x_n\}$, there is in $\mathcal{A}$ a monochromatic subset of $\{x_1, \dots, x_n\}$.*

Proof. Assume, towards a contradiction, that there is for each n a k-coloring c_n of the set $\{x_1, \dots, x_n\}$ with no monochromatic set in $\mathcal{A}$. Define a k-coloring of the set $X = \{x_1, x_2, \dots\}$ as follows:

(1) Choose a color i_1 such that the set $I_1 = \{\, n \in \mathbb{N} : c_n(x_1) = i_1 \,\}$ is infinite.
(2) Choose a color i_2 such that the set $I_2 = \{\, n \in I_1 : n \geq 2,\ c_n(x_2) = i_2 \,\}$ is infinite.
(3) By induction, for each $m > 1$ choose a color i_m such that the set

$$\begin{aligned} I_m &= \{\, n \in I_{m-1} : n \geq m,\ c_n(x_m) = i_m \,\} \\ &= \{\, n : n \geq m,\ c_n(x_1) = i_1, c_n(x_2) = i_2, \dots, c_n(x_m) = i_m \,\} \end{aligned}$$

is infinite.

Define a k-coloring of X by

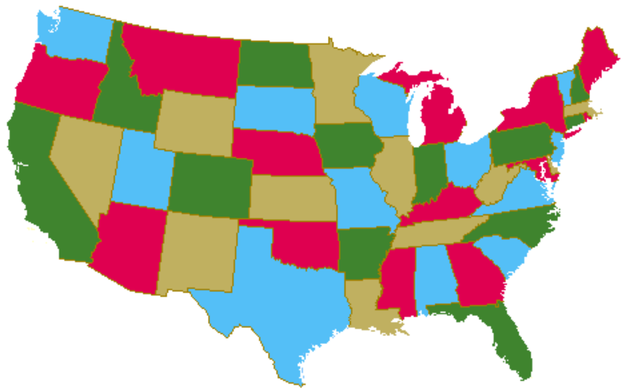
$$c(x_m) := i_m$$

for all m. By the premise of the theorem, there is a set $F \in \mathcal{A}$ that is monochromatic for the coloring c. Since the set F is finite, there is m such that $F \subseteq \{x_1, \dots, x_m\}$. Fix a natural number $n \in I_m$. Then the coloring c_n is defined on the elements $x_1, \dots, x_m$, and agrees with the coloring c there. It follows that the set F, which belongs to the family $\mathcal{A}$, is monochromatic for c_n, a contradiction. □

To illustrate the compactness theorem, we provide an amusing application. Assume that we have a (real, or imaginary) map of states, and we are interested in coloring each state region in a way that no neighboring states (sharing a border that is more than a point) have the same color. There are several formal restrictions on the map: That it is embedded in the plane, and that the state regions are "continuous", but the intuitive concept will suffice for our purposes.

What is the minimal number of colors necessary to color a map? The Google Chrome logo forms a minimal example of a map that cannot be colored with fewer than *four* colors:

On the other hand, even the state map of USA does not necessitate the use of more than four colors:



A nineteenth century conjecture, asserting that four colors suffice to color any map, was only proved in 1976 (by Kenneth Appel and Wolfgang Haken), by reducing the problem to fewer than 2,000 special cases, and checking them all on a computer. There is still no proof that does not necessitate the consideration of hundreds of cases, and we will of course not attempt at providing such a proof here. However, we will show that the formally stronger, infinite version of this theorem can be deduced from it.

THEOREM 2.2. *Every infinite state map can be colored with four colors, such that states sharing a border have different colors.*

PROOF. We first observe that the number of states in every map is countable. Indeed, think of the map as embedded in $\mathbb{R}^2$. Inside each state, choose a point (q_1, q_2) such that q_1 and q_2 are both rational numbers. The number of states is equal to the number of chosen points, which is not grater than the total number of points in $\mathbb{Q}^2$, which is countable.

Let $X = \{x_1, x_2, \dots\}$ be the set of states on the map. Let $\mathcal{A}$ be the set of neighboring pairs of states, that is, $\{x_i, x_j\} \in \mathcal{A}$ if and only if the states x_i and x_j share a border. Assume, towards a contradiction, that for each 4-coloring of X, there are neighboring states of the same color, that is, there is a monochromatic $\{x_i, x_j\} \in \mathcal{A}$. By the Compactness Theorem, there is n such that, for each 4-coloring of the finite map $\{x_1, \dots, x_n\}$, there is a monochromatic set $\{x_i, x_j\} \in \mathcal{A}$. But this contradicts the *finite* Four Color Theorem. $\square$

We now establish a finite version of Ramsey's Theorem. In the case $d = 2$, the theorem asserts that every large enough k-colored complete graph has a large complete monochromatic subgraph.

THEOREM 2.3 (Finite Ramsey Theorem). *Let k, m and d be natural numbers. There is n such that, for each k-coloring of the set $[\{1, \dots, n\}]^d$, there is a set $A \subseteq \{1, \dots, n\}$ of cardinality m such that the set $[A]^d$ is monochromatic.*

PROOF. Write $[\mathbb{N}]^d = \{x_1, x_2, \dots\}$. Let

$$\mathcal{A} = \left\{ [A]^d : A \subseteq \mathbb{N},\ |A| = m \right\}.$$

Every element of $\mathcal{A}$ is a finite subset of $\{x_1, x_2, \dots\}$.

Ramsey's Theorem asserts that for each k-coloring of $\{x_1, x_2, \dots\}$, there is an infinite set $B \subseteq \mathbb{N}$ such that the set $[B]^d$ is monochromatic. In particular, if we fix a subset $A \subseteq B$ of cardinality m, the element $[A]^d$ of the family $\mathcal{A}$ is monochromatic.

By the Compactness Theorem, there is a natural number N such that, for each k-coloring of $\{x_1, \dots, x_N\}$, there is in $\mathcal{A}$ a monochromatic subset of $\{x_1, \dots, x_N\}$. Let n be the largest element appearing in any x_i, formally $n = \max(x_1 \cup \cdots \cup x_N)$. Then

$$\{x_1, \ldots, x_N\} \subseteq [\{1, \ldots, n\}]^d,$$

and thus, for each k-coloring of the set $[\{1, \ldots, n\}]^d$, there is a monochromatic element $[A]^d$ in $\mathcal{A}$. $\square$

Ramsey's Theorem has numerous applications in mathematics and theoretical computer science. We will present here a beautiful application in number theory.

3. Fermat's Last Theorem and Schur's Theorem

In contrast to the Pythagorean Theorem, *Fermat's Last Theorem* asserts that, for $n > 2$ the equation

$$x^n + y^n = z^n$$

has no solution over the natural numbers. Fermat has stated this assertion without proof, and a proof was discovered only many generations later. The story of this theorem and its immensely complicated proof constitutes the topic of a best-selling book.

Long before Fermat's Last Theorem was proved, Issai Schur considered the problem whether Fermat's Equation has solutions *modulo a prime number*. One might hope that solving this problem for large enough prime numbers (and fixed n) may shed light on Fermat's assertion. Working with large primes also eliminates the following trivial obstacle.

Exercise 3.1. Prove that, for each prime number p, there is n such that the equation

$$x^n + y^n = z^n \pmod{p}$$

has no nontrivial solution $x, y, z \neq 0 \pmod{p}$.
Hint: Consider *Fermat's Little Theorem*: For each $a \in \{0, 1, \ldots, p-1\}$, we have $a^p = a \pmod{p}$.

Theorem 3.2 (Schur). *For each large enough prime number p, the equation $x^n + y^n = z^n \pmod{p}$ has a solution with $x, y, z \neq 0 \pmod{p}$.*

The proof of Schur's Theorem uses the following interesting theorem.

Theorem 3.3 (Schur's Coloring Theorem). *For each finite coloring of $\mathbb{N}$, there are natural numbers x,y and z of the same color such that $x+y=z$. In other words, the equation $x+y=z$ has a monochromatic solution.*

Proof. Let c be a k-coloring of $\mathbb{N}$. Define a k-coloring χ of $[\mathbb{N}]^2$ by

$$\chi(\{i, j\}) = c(j - i)$$

for all $i < j$. By Ramsey's Theorem, there is an infinite set $A \subseteq \mathbb{N}$ such that $[A]^2$ is monochromatic for χ. Let $i, j, m \in A$ be such that $i < j < m$. By the definition of χ, we have $c(j-i) = c(m-j) = c(m-i)$, and we obtain a monochromatic solution

$$\underbrace{m-j}_{x} + \underbrace{j-i}_{y} = \underbrace{m-i}_{z}. \quad \square$$

Exercise 3.4. Show, using the proof of Schur's Coloring Theorem, that the equation $x+y = z$ has infinitely many monochromatic solutions.

Corollary 3.5. *For every number of colors k, there is n such that, for each k-coloring of $\{1, \ldots, n\}$, the equation $x + y = z$ has a monochromatic solution.*

Exercise 3.6. Prove Corollary 3.5, using the Compactness Theorem.

Proof of Theorem 3.2. Consider the finite field $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$, with addition and multiplication modulo p. A fundamental theorem asserts that for each finite field $\mathbb{F}$, there is an element $g \in \mathbb{F}$ such that every nonzero element of $\mathbb{F}$ is a power of g. Let $0 \neq g \in \mathbb{Z}_p$ be such that $\{g, g^2, \ldots, g^{p-1}\} = \mathbb{Z}_p \setminus \{0\}$.

Define a coloring of the set $\{1, \ldots, p-1\}$ as follows: For each element $r \in \{1, \ldots, p-1\}$, there is a unique number $m \in \{1, \ldots, p-1\}$ such that $g^m = r$. Set $c(r) := m \bmod n$.

If the prime number p is large enough, then by Schur's Coloring Theorem there are numbers $x, y, z \in \{1, \ldots, p-1\}$, of the same color, such that $x + y = z$ over $\mathbb{N}$ and, in particular, over $\mathbb{Z}_p$. Continue the argument in $\mathbb{Z}_p$. Write

$$x = g^{nt_1+i}, y = g^{nt_2+i}, z = g^{nt_3+i}.$$

Then

$$g^{nt_1+i} + g^{nt_2+i} = g^{nt_3+i}.$$

Divide by g^i, to obtain

$$(\underbrace{g^{t_1}}_{x})^n + (\underbrace{g^{t_2}}_{y})^n = (\underbrace{g^{t_3}}_{z})^n.$$

We have thus found nonzero elements $x, y, z \in \mathbb{Z}_p$ such that $x^n + y^n = z^n \pmod p$. $\square$

Remark 3.7. The proof of Schur's Theorem shows that for each n, in every large enough finite field $\mathbb{F}$, the equation $x^n + y^n = z^n$ has a nontrivial solution.

The following exercises can be solved by modifying the above arguments.

Exercise 3.8. Let n be a natural number. Prove that for each large enough prime number p, the equation $x^n + y^n + z^n = w^n \pmod p$ has a solution with $x, y, z, w \neq 0 \pmod p$.

Exercise 3.9. Let G be a group with at least 6 elements. Prove that for each 2-coloring of G, there are nonidentity elements $a, b, c \in G$ of the same color, such that $ab = c$.

4. Comments for Chapter 1

Ramsey's Theorem is proved in his paper *On a problem of formal logic*, Proceedings of the London Mathematical Society 30 (1928), 264–286. Theorem 3.2 was first proved by Leonard E. Dickson, *On the Last Theorem of Fermat*, Quarterly Journal of Pure and Applied Mathematics, 1908. The proof provided here, via Schur's Coloring Theorem (Theorem 3.3), is due to Issai Schur, *Über die Kongruenz* $x^m + y^m = z^m \pmod p$, Jahresbericht der Deutschen Mathematiker–Vereinigung, 1916.

When a coloring theorem guarantees the existence of an infinite monochromatic set, it may be strictly stronger than its finite version. It follows from Ramsey's Theorem that, for all d, k, and m:

> There is n such that, for each k-coloring of $[\{m, m+1, \ldots, n\}]^d$ there is a set $A \subseteq \{m, m+1, \ldots, n\}$ such that $|A| > \min A$ and $[A]^d$ is monochromatic.

Exercise 4.1. Prove the last assertion.

The finite Ramsey Theorem is provable in Peano Arithmetic (the basic axiomatic system for number theory). Paris and Harrington, that identified the above consequence of Ramsey's theorem, proved that it is *unprovable* in Peano Arithmetic. This was the first natural statement in the language of Peano Arithmetic that is true but not provable. The mere existence of such statements follows from Gödel's celebrated Incompleteness Theorem. This topic is covered in Section 6.3 of the Graham–Spencer–Rothschild classic book *Ramsey Theory*.

The finitary theorems may be thought of as shadows of their infinite counterpart. In general, the question how large should the finite colored set be to guarantee a monochromatic set as desired is wide open. For example, let r_m be the minimal natural number n such that the Finite Ramsey Theorem holds for $k = 2$ colors, dimension $d = 2$, and $|A| = m$. It follows from the proof of Ramsey's Theorem that $r_3 = 6$. It is known that $r_4 = 18$. But in general, despite great efforts, only weak bounds are available for the remaining numbers r_m. According to Joel Spencer (*Ten Lectures on the Probabilistic Method*, SIAM, 1994),

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of r_5 or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for r_6. In that case, he believes, we should attempt to destroy the aliens.

The Wikipedia entry *Ramsey's theorem* provides additional details on this direction of research.

CHAPTER 2

Ultrafilters, topology and compactness

In the remainder of the book, we will prove coloring theorems that have no simple elementary proofs. We will exploit the interplay between algebra and topology. This chapter provides the foundations of this method. Readers familiar with these foundations may find it sufficient to skim this chapter briefly, and proceed to the next chapter.

1. Filters and ultrafilters

For a set X, let $P(X) = \{ B : B \subseteq X \}$, the family of all subsets of X. We will be interested in *filters* on X, families of sets which contain, intuitively speaking, "very large" subsets of X. This is the motivation behind the following definition.

Definition 1.1. A *filter* on a set X is a family $\mathcal{F}$ of subsets of X such that, for all $A, B \subseteq X$:

(1) $X \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
(2) If $B \supseteq A \in \mathcal{F}$, then $B \in \mathcal{F}$.
(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Inductively, for all $A_1, \dots, A_n$ in a filter $\mathcal{F}$ we have that $A_1 \cap \cdots \cap A_n \in \mathcal{F}$. By the second property, the first property may be restated as $\emptyset \subsetneq \mathcal{F} \subsetneq P(X)$.

Definition 1.2. The *principal filter* at an element $x \in X$ is the family

$$\mathcal{F}_x = \{ A \subseteq X : x \in A \} .$$

The principal filter is a "dictatorship", in the sense that the decision whether a set is large or not is determined by x only.

Exercise 1.3. Prove that a filter $\mathcal{F}$ is principal if and only if it contains a singleton (a single-element set).

Example 1.4. Let X be an infinite set. A set $A \subseteq X$ is *cofinite* (in X) if its complement, $A^{\mathsf{c}} := X \setminus A$, is finite. The *cofinite sets filter* is the family $\mathcal{F}$ of all cofinite subsets of X. Since X is infinite, the cofinite sets filter is indeed a filter.

Lemma 1.5. *Let $\mathcal{A}$ be a nonempty family of subsets of X, such that every intersection of finitely many elements of $\mathcal{A}$ is nonempty. Then the closure of $\mathcal{A}$ under taking finite intersections and supersets,*

$$\langle \mathcal{A} \rangle := \{ B \subseteq X : \exists A_1, \dots, A_k \in \mathcal{A},\ A_1 \cap \cdots \cap A_k \subseteq B \} ,$$

is a filter on X. This is the smallest filter containing $\mathcal{A}$. We will call it the filter generated by *$\mathcal{A}$.*

Proof. It is easy to verify that the family $\langle \mathcal{A} \rangle$ satisfies each property of a filter. Every filter containing $\mathcal{A}$ must contain every intersection of finitely many members of $\mathcal{A}$, and thus also every set containing such an intersection. □

Definition 1.6. A filter $\mathcal{F}$ on a set X is an *ultrafilter* if for each $A \subseteq X$, we have that $A \in \mathcal{F}$ or $A^{\mathsf{c}} \in \mathcal{F}$.

EXAMPLE 1.7. For each $x \in X$, the principal filter $\mathcal{F}_x$ is an ultrafilter.

EXERCISE 1.8. Prove that if $\mathcal{F}$ is an ultrafilter on a (finite or infinite) set X, and $\mathcal{F}$ has some finite member, then $\mathcal{F}$ is principal. Conclude that every ultrafilter on a finite set X is principal.

We will only consider the case where X is infinite.

EXAMPLE 1.9. The cofinite sets filter on an infinite set X is not an ultrafilter, since X can be partitioned into two infinite, disjoint sets, and none of these sets is cofinite.

EXERCISE 1.10. Let $\mathcal{F}$ be an ultrafilter on X. Prove the following assertions:

(1) If $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.
(2) If $A_1 \cup \cdots \cup A_k \in \mathcal{F}$, then there is $i \leq k$ such that $A_i \in \mathcal{F}$.

We obtain the following connection between ultrafilters and colorings.

COROLLARY 1.11. *Let A be a member of an ultrafilter $\mathcal{F}$. For each finite coloring of A, there is in $\mathcal{F}$ a monochromatic subset of A.*

PROOF. Let $c\colon A \to \{1, \ldots, k\}$ be a coloring. For each $i \in \{1, \ldots, k\}$, let $A_i = c^{-1}(i)$, the set of elements in A of color i. Then

$$A_1 \cup \cdots \cup A_k = A \in \mathcal{F}.$$

By Exercise 1.9, there is i such that $A_i \in \mathcal{F}$. □

We will use Zorn's Lemma to establish the existence of nonprincipal ultrafilters. Zorn's Lemma provides a sufficient condition for the existence of maximal elements. Our interest in maximal elements is explained by the following lemma.

LEMMA 1.12. *A filter $\mathcal{F}$ on a set X is an ultrafilter if and only if it is a maximal filter on X, that is, there is no filter $\mathcal{F}'$ on X with $\mathcal{F} \subsetneq \mathcal{F}'$.*

PROOF. ($\Rightarrow$) Assume that there is a filter $\mathcal{F}'$ properly extending $\mathcal{F}$. Let $A \in \mathcal{F}' \setminus \mathcal{F}$. As $\mathcal{F}$ is an ultrafilter, we have that $A^{\mathsf{c}} \in \mathcal{F}$. Thus, A and A^{c} are both in $\mathcal{F}'$; a contradiction.

($\Leftarrow$) Let $A \subseteq X$, and assume that $A, A^{\mathsf{c}} \notin \mathcal{F}$. If there was a set $B \in \mathcal{F}$ with $B \subseteq A^{\mathsf{c}}$, then $A^{\mathsf{c}} \in \mathcal{F}$; a contradiction. Thus, every element B of $\mathcal{F}$ intersects A. It follows that every intersection of finitely many elements of $\mathcal{F} \cup \{A\}$ is nonempty, and the generated filter $\langle \mathcal{F} \cup \{A\} \rangle$ properly extends $\mathcal{F}$ (since $A \notin \mathcal{F}$). This contradicts the maximality of $\mathcal{F}$. □

The following special form of Zorn's Lemma suffices for our needs. Let $\mathcal{A}$ be a family of sets. An element $A \in \mathcal{A}$ is *maximal* if there is no $B \in \mathcal{A}$ with $A \subsetneq B$. A *chain* in $\mathcal{A}$ is a family $\{\, A_\alpha : \alpha \in I \,\}$ of elements of $\mathcal{A}$ such that for all $\alpha, \beta \in I$, we have that $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$.

LEMMA 1.13 (Zorn's Lemma). *Let $\mathcal{A}$ be a nonempty family of sets, with the property that for each chain $\{\, A_\alpha : \alpha \in I \,\}$ in $\mathcal{A}$, we have that $\bigcup_{\alpha \in I} A_\alpha \in \mathcal{A}$. Then there is a maximal element in $\mathcal{A}$.*

LEMMA 1.14 (Ultrafilter Theorem). *Every filter $\mathcal{F}$ on a set X extends to an ultrafilter on X.*

PROOF. Let $\mathcal{A}$ be the set of all filters $\mathcal{F}'$ on X with $\mathcal{F} \subseteq \mathcal{F}'$. $\mathcal{A}$ is nonempty since $\mathcal{F} \in \mathcal{A}$. For each chain of filters in $\mathcal{F}$, their union is a filter and is in $\mathcal{A}$. By Zorn's Lemma, there is a maximal element $\mathcal{F}'$ in P. By Lemma 1.11, $\mathcal{F}'$ is an ultrafilter. □

EXERCISE 1.15. Prove that the union of a chain of filters is a filter.

For the proof of the following corollary, recall that, by Exercise 1.7, every nonprincipal ultrafilter on a set X must contain all cofinite subsets of X.

Corollary 1.16. *Let X be an infinite set.*

(1) *There is a nonprincipal ultrafilter on X.*
(2) *For each infinite set $A \subseteq X$, there is a nonprincipal ultrafilter $\mathcal{F}$ on X such that $A \in \mathcal{F}$.*
(3) *Let $\mathcal{A}$ be a family of subsets of X such that every intersection of finitely many members of $\mathcal{A}$ is infinite. Then there is a nonprincipal ultrafilter $\mathcal{F}$ on X such that $\mathcal{A} \subseteq \mathcal{F}$.*

Proof. (3) Let $\mathcal{C}$ be the filter of cofinite sets in X. Given finitely many elements of $\mathcal{A} \cup \mathcal{C}$, the intersection of those from $\mathcal{C}$ is cofinite, and the intersection of those from $\mathcal{A}$ is infinite. Thus, the intersection of them all is infinite. Let $\langle \mathcal{A} \cup \mathcal{C} \rangle$ be the filter on X generated by $\mathcal{A} \cup \mathcal{C}$. Extend this filter to an ultrafilter $\mathcal{F}$ on X. Then $\mathcal{A} \subseteq \mathcal{F}$.

For each finite set $F \subseteq X$, its complement F^{c} is in $\mathcal{C}$ and thus in $\mathcal{F}$. Thus, $F \notin \mathcal{C}$. In particular, for each $x \in X$, we have that $\{x\} \notin \mathcal{F}$, and therefore $\mathcal{F} \neq \mathcal{F}_x$.

(2) Take $\mathcal{A} = \{A\}$ in (3).

(1) Take $\mathcal{A} = \emptyset$ in (3). □

2. Excursion: An ultrafilter proof of Ramsey's Theorem

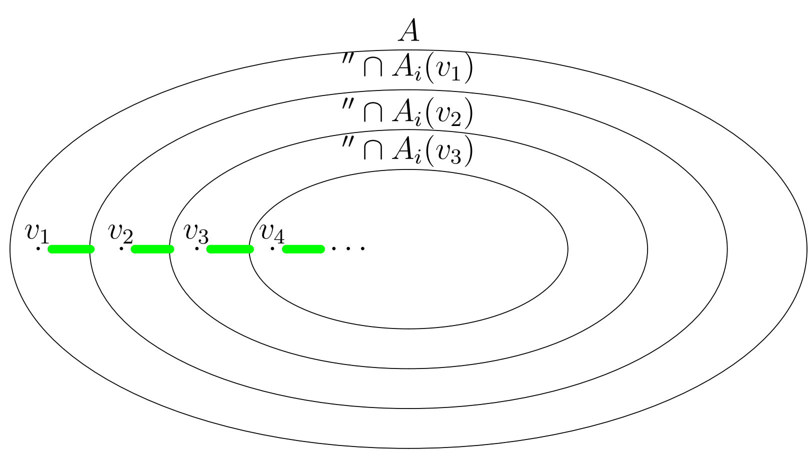
The following proof is not necessarily simpler than Ramsey's original, but it has a surprising feature: Using an ultrafilter, we can predict the color of the final monochromatic graph before actually searching for it.

Ultrafilter proof of Ramsey's Theorem. Let $\mathcal{F}$ be a nonprincipal ultrafilter on the set V of graph vertices. Let $c\colon [V]^2 \to \{1, \dots, k\}$ be a k-coloring of the complete graph. For each $v \in V$ and each $i \in \{1, \dots, k\}$, let $A_i(v)$ be the set of all vertices connected to v by an edge of color i. Then

$$V \setminus \{v\} = A_1(v) \cup \cdots \cup A_k(v).$$

As the ultrafilter $\mathcal{F}$ is nonprincipal, $V \setminus \{v\} \in \mathcal{F}$, and thus there is (a unique) i with $A_i(v) \in \mathcal{F}$. Define $\chi(v) := i$. We obtain a k-coloring χ of V.

As $\mathcal{F}$ is an ultrafilter, there is a monochromatic set $A \in \mathcal{F}$ for χ. Let i be the color of this set. We will show that there is an infinite, complete monochromatic subgraph *of color i*:



Take

$$\begin{array}{rcl} v_1 & \in & A, \\ v_2 & \in & A \cap A_i(v_1), \\ v_3 & \in & A \cap A_i(v_1) \cap A_i(v_2), \\ & \vdots & \end{array}$$

This can be done since, in each stage, we choose from the intersection of finitely many elements of a filter. Being in the filter, such an intersection is never empty.

The complete graph with vertex set $\{ v_n : n \in \mathbb{N} \}$ is monochromatic, of color i: For all $n < m$, we have that $v_m \in A_i(v_n)$, and thus $c(v_n, v_m) = i$. □

The ultrafilter proof of Ramsey's Theorem can be explained, intuitively, as follows. Fix an ultrafilter on V. Say that a set of vertices forms a "vast majority" if it is in the ultrafilter. Then each vertex has a *preferred color* such that its edges to the vast majority of its neighbors have that color. Taking a color that is preferred by the vast majority of the vertices, it is easy to construct an infinite, complete graph of that color.

What about the finite Ramsey Theorem? Ultrafilters are not of direct use for studying finite objects, since ultrafilters containing finite sets are principal. But they do provide an alternative proof of the Compactness Theorem, from which the finite Ramsey Theorem follows. As a bonus, the proof gives a stronger result, where the colored "board" X is not assumed to be countable. The proof of the countable case, which suffices for our needs, is provided first. We then explain how it extends to the uncountable case. Readers not familiar with ordinal and cardinal numbers may skip this part of the proof.

THEOREM 2.1 (Full Compactness). *Let X be an infinite set, and $\mathcal{A}$ be a family of finite subsets of X. Assume that for each k-coloring of X there is a monochromatic set in $\mathcal{A}$. Then there is a finite set $F \subseteq X$ such that for each k-coloring of F there is a monochromatic set in $\mathcal{A}$.*

PROOF. *The countable case.* Enumerate $X = \{ x_n : n \in \mathbb{N} \}$, and assume that there is no finite set as required in the theorem. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$.

For each n, fix a coloring $c_n \colon \{x_1, \dots, x_n\} \to \{1, \dots, k\}$ with no monochromatic set in $\mathcal{A}$. We define a k-coloring c of X by assigning, to each $x \in X$, the color assigned to x by the majority—with respect to the ultrafilter $\mathcal{F}$—of colorings c_n: Fix a number $m \in \mathbb{N}$. Since

$$\{ n : n \geq m, c_n(x_m) = 1 \} \cup \cdots \cup \{ n : n \geq m, c_n(x_m) = k \} = \{m, m+1, \dots\} \in \mathcal{F},$$

there is $i_m \in \{1, \dots, k\}$ such that

$$\{ n : n \geq m, c_n(x_m) = i_m \} \in \mathcal{F}.$$

Define $c(x_m) = i_m$.

By the premise of the theorem, there is a set $\{x_{m_1}, \dots, x_{m_N}\} \in \mathcal{A}$ that is monochromatic for the coloring c, say of color i. Then

$$\{ n : n \geq m_1, c_n(x_{m_1}) = i \} \cap \cdots \cap \{ n : n \geq m_N, c_n(x_{m_N}) = i \}$$

is an intersection of finitely many elements of $\mathcal{F}$, and is thus nonempty. Let n be an element in this intersection. Then the set $\{x_{m_1}, \dots, x_{m_N}\}$, which is in $\mathcal{A}$, is monochromatic for c_n; a contradiction.

The uncountable case. The treatment of this case is a straightforward modification of the countable case. Assume that the theorem fails, and let X be a counterexample of minimal cardinality κ. Enumerate $X = \{ x_\alpha : \alpha < \kappa \}$. Since finite intersections of sets of the form

$$[\alpha, \kappa) := \{ \gamma < \kappa : \gamma \geq \alpha \},$$

for $\alpha < \kappa$, are nonempty, there is an ultrafilter $\mathcal{F}$ on κ containing these sets.

For each $\alpha < \kappa$, the cardinality of the set $\{ x_\beta : \beta \leq \alpha \}$ is smaller than κ. By the minimality of κ, there is a coloring $c_\alpha \colon \{ x_\beta : \beta \leq \alpha \} \to \{1, \dots, k\}$ with no monochromatic set in $\mathcal{A}$ (see Exercise 2.2). Since

$$\{ \gamma < \kappa : \gamma \geq \alpha, c_\gamma(x_\alpha) = 1 \} \cup \cdots \cup \{ \gamma < \kappa : \gamma \geq \alpha, c_\gamma(x_\alpha) = k \} = [\alpha, \kappa) \in \mathcal{F},$$

there is $i_\alpha \in \{1, \dots, k\}$ such that $\{ \gamma < \kappa : \gamma \geq \alpha, c_\gamma(x_\alpha) = i_\alpha \} \in \mathcal{F}$. Define $c(x_\alpha) = i_\alpha$.

By the premise of the theorem, there is a set $\{x_{\alpha_1}, \dots, x_{\alpha_N}\} \in \mathcal{A}$ that is monochromatic for the coloring c, say of color i. Then

$$\{ \gamma < \kappa : \gamma \geq \alpha_1, c_\gamma(x_{\alpha_1}) = i \} \cap \cdots \cap \{ \gamma < \kappa : \gamma \geq \alpha_N, c_\gamma(x_{\alpha_N}) = i \} \in \mathcal{F}.$$

Let γ be an element in this intersection. Then the set $\{x_{\alpha_1}, \dots, x_{\alpha_N}\}$, which is in $\mathcal{A}$, is monochromatic for c_γ; a contradiction. $\square$

Exercise 2.2. Prove the assertion in the proof of Theorem 2.1, that for each $\alpha < \kappa$ there is a k-coloring of the set $\{ x_\beta : \beta \leq \alpha \}$ with no monochromatic set in $\mathcal{A}$.

3. Topological spaces

Definition 3.1. A *topology* on a set X is a family τ of subsets of X with the following properties:

1. $\emptyset, X \in \tau$.
2. For all $U, V \in \tau$, also $U \cap V \in \tau$.
3. For each family $\{ U_\alpha : \alpha \in I \} \subseteq \tau$, also $\bigcup_{\alpha \in I} U_\alpha \in \tau$.
4. The *Hausdorff property*: For all distinct $x, y \in X$ there are disjoint sets $U, V \in \tau$ such that $x \in U, y \in V$.

A *topological space* is a space equipped with a topology τ. The sets U is the topology τ of a topological space are called *open sets*.

Remark 3.2. Topological spaces are often defined without requesting them to have the Hausdorff property. In such cases, topological spaces satisfying the above four requirements are called *Hausdorff spaces* or *T_2 spaces*. In this book, all considered spaces are Hausdorff, and thus we do not make this distinction.

Definition 3.3. A *neighborhood* U of a point x in a topological space X is an open set U such that $x \in U$.

Definition 3.4. The closure $\overline{A}$ of a set $A \subseteq X$ in a topological space X is the set of all points x such that every neighborhood of x intersects A. A set $A \subseteq X$ is *closed* if $\overline{A} = A$.

Exercise 3.5. Prove that a set A in a topological space is closed if and only if its complement A^{c} is open.

Definition 3.6. We say that $\mathcal{F}$ is a *filter in* X if $\mathcal{F}$ is a filter on some subset of X. Similarly, an *ultrafilter in* X is an ultrafilter on some subset of X.

The family of all neighborhoods of a point x is closed under finite intersections, and therefore its closure under taking supersets is a filter, called the *neighborhood filter* at x and denoted $\mathcal{N}_x$. The neighborhood filter $\mathcal{N}_x$ is "concentrated" around the point x. The following definition generalizes this property.

DEFINITION 3.7. Let $\mathcal{F}$ be a filter in a topological space X. A point $x \in X$ is a *limit point* of $\mathcal{F}$ if for each neighborhood U of x there is in $\mathcal{F}$ some subset of U.

LEMMA 3.8. *A filter in a topological space X can have at most one limit point*

PROOF. Assume that x and y are distinct limit points of a filter $\mathcal{F}$ in X. By the Hausdorff property, there are disjoint neighborhoods U and V of x and y, respectively. Since x and y are limit points of $\mathcal{F}$, there are sets $A, B \in \mathcal{F}$ such that $A \subseteq U$ and $B \subseteq V$. Then $\emptyset = A \cap B \in \mathcal{F}$; a contradiction. □

DEFINITION 3.9. Let $\mathcal{F}$ be a filter in a topological space X, and x be a point in X. The filter $\mathcal{F}$ *converges to* x (in symbols: $\lim \mathcal{F} = x$ or $\mathcal{F} \to x$) if x is a limit point of X.

For example, for each $x \in X$ we have that $\lim \mathcal{N}_x = x$.

Let $x_1, x_2, \dots \in X$ be distinct points. The standard definition for $\lim_{n\to\infty} x_n = x$ is: For each neighborhood U of x, we have that $x_n \in U$ for all but finitely many n. Equivalently, we can say that the filter of cofinite subsets on $\{x_1, x_2, \dots\}$ converges to x. The reason why the ordinary definition is insufficient for our purposes is that, for the most important topological space considered in this book, the following theorem fails if we replace "filter" by "sequence".

THEOREM 3.10. *Let X be a topological space and $A \subseteq X$. A point x is in $\overline{A}$ if and only if some filter $\mathcal{F}$ in A converges to x.*

PROOF. ($\Rightarrow$) Let $\mathcal{N}_x$ be the neighborhood filter at x. As every neighborhood of x intersects A, the set $\mathcal{F} = \{\, N \cap A : N \in \mathcal{N}_x \,\}$ is a filter. Clearly, $\mathcal{F} \to x$.

($\Leftarrow$) Every neighborhood of x contains an element of $\mathcal{F}$, and this element of $\mathcal{F}$ is nonempty and contained in A. □

DEFINITION 3.11. Let X and Y be topological spaces, $f\colon X \to Y$, and $a \in X$.

(1) f is *continuous at a* if for each neighborhood V of $f(a)$ there is a neighborhood U of a such that $f(U) \subseteq V$.
(2) f is *continuous* if f is continuous at all points of X.

EXERCISE 3.12. Prove that, for topological spaces X and Y, a function $f\colon X \to Y$ is continuous if and only if for each open set V in Y, the set $f^{-1}(V)$ is open in X.

For a function $f\colon X \to Y$ and sets $A \subseteq X, B \subseteq Y$:

$$\begin{aligned} f(A) &:= \{\, f(a) : a \in A \,\} \\ f^{-1}(B) &:= \{\, x \in X : f(x) \in B \,\} \end{aligned}$$

In particular, for a filter $\mathcal{F}$ in X, we have that

$$f(\mathcal{F}) = \{\, f(A) : A \in \mathcal{F} \,\}.$$

LEMMA 3.13. *Let $f\colon X \to Y$.*

(1) *For each filter $\mathcal{F}$ in X, the family $f(\mathcal{F})$ is a filter in Y.*
(2) *For each ultrafilter $\mathcal{F}$ in X, the family $f(\mathcal{F})$ is an ultrafilter in Y.*

EXERCISE 3.14. Prove Lemma 3.13.

EXERCISE 3.15. Prove that a function $f\colon X \to Y$ is continuous at a if and only if for each filter $\mathcal{F}$ in X converging to a, we have that $f(\mathcal{F}) \to f(a)$.

4. Compact spaces

Definition 4.1. A *cover* of a set X is a family $\mathcal{U} = \{ U_\alpha : \alpha \in I \}$ of subsets of X such that $X = \bigcup_{\alpha\in I} U_\alpha$. An *open cover* of a topological space X is a cover $\mathcal{U}$ of X such that all elements of $\mathcal{U}$ are open in X.

A family $\mathcal{V}$ is a *subcover* of a cover $\mathcal{U}$ of X if $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{V}$ is a cover of X. A *compact space* is a topological space X such that every open cover of X has a finite subcover.

Definition 4.2. Let X be a topological space with topology τ_X, and let $Y \subseteq X$. The *induced topology* on Y is the family

$$\tau_Y = \{ U \cap Y : U \in \tau_X \}.$$

A *subspace* of a topological space X is a subset Y of X, equipped with the induced topology.

Subsets of topological spaces will always be considered as the subspaces equipped with the induced topology.

Exercise 4.3. Prove that a subset Y of a topological space X is a topological space.

Lemma 4.4. *Every closed set in a compact space is compact.*

Proof. Let X be a compact space. Let $A \subseteq X$ be closed. Let $\{ U_\alpha \cap A : \alpha \in I \}$ be an open cover of A, with each U_α open in X. A^{c} is open in X, and thus the family $\{ U_\alpha \cup A^{\mathsf{c}} : \alpha \in I \}$ is an open cover of X. Let $\{U_{\alpha_1} \cup A^{\mathsf{c}}, \dots, U_{\alpha_n} \cup A^{\mathsf{c}}\}$ be a finite subcover. Then $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite cover of A. $\square$

Exercise 4.5. Let X be a compact space, Y a topological space, and $f\colon X \to Y$ a continuous function. Prove that the image $f(X)$ of f, as a subspace of Y, is compact. In particular, if f is surjective, then Y is compact.

Theorem 4.6. *A topological space X is compact if and only if every ultrafilter in X is convergent.*

Proof. ($\Rightarrow$) Let $\mathcal{F}$ be an ultrafilter in X without a limit point: Each $x \in X$ has a neighborhood U_x containing no element of $\mathcal{F}$. As X is compact, the open cover $\{ U_x : x \in X \}$ has a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Fix a set $A \in \mathcal{F}$. As $A \subseteq X = U_{x_1} \cup \dots \cup U_{x_n}$, we have that

$$(A \cap U_{x_1}) \cup (A \cap U_{x_2}) \cup \dots \cup (A \cap U_{x_n}) = A \in \mathcal{F},$$

and thus there is $i \in \{1, \dots, n\}$ such that the subset $A \cap U_{x_i}$ of U_{x_i} is in $\mathcal{F}$; a contradiction.

($\Leftarrow$) Assume that X is not compact. Let $\{ U_\alpha : \alpha \in I \}$ be an open cover of X with no finite subcover. Then every intersection of finitely many elements of the family $\mathcal{A} = \{ U_\alpha^{\mathsf{c}} : \alpha \in I \}$ is nonempty. Indeed, for all $\alpha_1, \dots, \alpha_n \in I$, since $U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \neq X$, we have that

$$U_{\alpha_1}^{\mathsf{c}} \cap \dots \cap U_{\alpha_n}^{\mathsf{c}} = (U_{\alpha_1} \cup \dots \cup U_{\alpha_n})^{\mathsf{c}} \neq \emptyset.$$

Extend the filter $\langle \mathcal{A} \rangle$ generated by $\mathcal{A}$ to an ultrafilter $\mathcal{F}$ on X. Let x be the limit point of $\mathcal{F}$. Pick $\alpha \in I$ with $x \in U_\alpha$. Take $A \in \mathcal{F}$ such that $A \subseteq U_\alpha$. Then A is disjoint of U_α^{c}, which is in $\mathcal{F}$. We therefore have two disjoint sets in the ultrafilter $\mathcal{F}$; a contradiction. $\square$

It is often convenient to describe a topology on a space by specifying "basic" open sets only.

Definition 4.7. Let X be a topological space. A family $\mathcal{B}$ of open subsets of X is a *basis* for the topology of X if every open subset of X is a union of elements of $\mathcal{B}$, explicitly:

(1) Every $U \in \mathcal{B}$ is an open set.
(2) For each open set U, there is a family $\{ U_\alpha : \alpha \in I \} \subseteq \mathcal{B}$ such that $\bigcup_{\alpha\in I} U_\alpha = U$.

The elements of a basis $\mathcal{B}$ are called *basic open* sets.

A basis can be used to *define* a topology on a set X, by declaring a set open if and only if it is a union of members of $\mathcal{B}$.

LEMMA 4.8. *Let X be a set, and let $\mathcal{B} = \{ B_\alpha : \alpha \in I \}$ be a family of subsets of X with the following properties:*

(1) *The union of all sets in $\mathcal{B}$ is X.*
(2) *For all $U, V \in \mathcal{B}$, we have that $U \cap V \in \mathcal{B}$.*
(3) *For all distinct $x, y \in X$ there are disjoint sets $U, V \in \mathcal{B}$ such that $x \in U, y \in V$.*

Then the family $\tau := \left\{ \bigcup_{\alpha \in J} B_\alpha : J \subseteq I \right\}$ of all unions of elements of $\mathcal{B}$ is a topology on X.

EXERCISE 4.9. Prove Lemma 4.8.

DEFINITION 4.10. Let X and Y be topological spaces. The *product topology* on the set $X \times Y$ is the one with basic open sets $U \times V$, for U open in X and V open in Y. In general, the basis of a product $X_1 \times X_2 \times \cdots \times X_n$ of n topological spaces consists of the sets $U_1 \times \cdots \times U_n$ with U_i open X_i for all $i = 1, \ldots, n$.

EXERCISE 4.11. Prove that the product topology is indeed a topology, that is, that the defined basis for this topology is indeed a basis for a topology.

In this book, product spaces are always considered as topological spaces with respect to the product topology. When considering a topological space X and a subspace Y of X, the term *open cover of* Y will be interpreted, for convenience only, as a cover of Y by sets open in X.

THEOREM 4.12. *Finite products of compact spaces are compact.*

PROOF. It suffices to prove that the product of two compact spaces is compact. Let X and Y be compact spaces. It suffices to prove that every ultrafilter in $X \times Y$ converges (Theorem 4.6). Let $\mathcal{F}$ be an ultrafilter in $X \times Y$. Consider the *projection functions*

$$\begin{array}{rcl} \pi_1 \colon X \times Y & \to & X \\ (x, y) & \mapsto & x \end{array}$$

and

$$\begin{array}{rcl} \pi_2 \colon X \times Y & \to & Y \\ (x, y) & \mapsto & y \end{array}$$

The projections $\pi_1(\mathcal{F})$ and $\pi_2(\mathcal{F})$ of the ultrafilter $\mathcal{F}$ in X and Y, respectively, are ultrafilters there, and thus converge to points $x \in X$ and $y \in Y$, respectively. We will show that $\mathcal{F}$ converges to the point $(x, y) \in X \times Y$.

Let U be an open set in $X \times Y$ with $(x, y) \in U$. Since U is a union of basic open sets, the point (x, y) lies in some basic open set $V \times W$ contained in U. As $\pi_1(x, y) = x \in V$, there is $A \in \mathcal{F}$ such that $\pi_1(A) \subseteq V$. Similarly for y, there is $B \in \mathcal{F}$ such that $\pi_2(B) \subseteq W$. Then the set $A \cap B$ is in $\mathcal{F}$, and $\pi_1(A \cap B) \subseteq V, \pi_2(A \cap B) \subseteq W$ and thus $A \cap B \subseteq V \times W \subseteq U$. $\square$

5. Excursion: Tychonoff's product theorem

DEFINITION 5.1. Let $\{ X_\alpha : \alpha \in I \}$ be a family of topological spaces. The *product* of this family is the set

$$\prod_{\alpha \in I} X_\alpha$$

of all sequences $(x_\alpha)_{\alpha\in I}$ with $x_\alpha \in X_\alpha$ for all $\alpha \in I$. A basis for a topology on $\prod_{\alpha\in I} X_\alpha$ consists of the sets

$$[\alpha_1,\dots,\alpha_n\,;U_{\alpha_1},\dots,U_{\alpha_n}] := \Big\{ (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha : x_{\alpha_i} \in U_{\alpha_i} \text{ for all } i = 1,\dots,n \Big\},$$

for $n \in \mathbb{N}$, $\alpha_1,\dots,\alpha_n \in I$, and U_{α_i} open in X_{α_i} for all $i = 1,\dots,n$.

The (Tychonoff) *product topology* is the topology on the product $\prod_{\alpha\in I} X_\alpha$ defined by this basis. The resulting topological space is called the *product space*.

An illustration of a basic open set in the product topology is provided in Figure 1. This is like slalom skiing. The sequence has to pass through all of the (finitely many) prescribed open sets. The sequence described by the green curve is in the open sets, since in each of the indices a_1, a_2, a_3, it passes through the requested open set (denoted by a blue interval). In contrast, the sequence described by the red curve (a heavy skier) does not belong to the open set, since it misses the open set at index a_2.

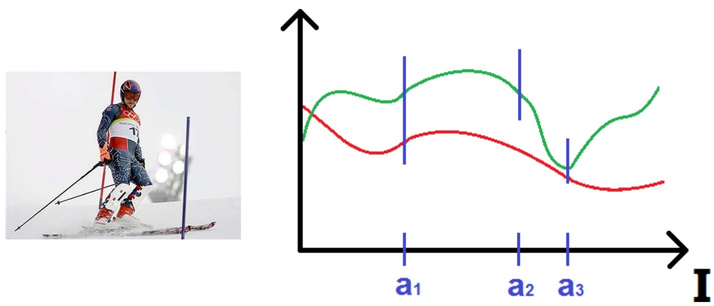


FIGURE 1. A slalom defining the basic open set $[a_1, a_2, a_3\,; U_{a_1}, U_{a_2}, U_{a_3}]$.

EXERCISE 5.2. Prove that, in the definition of the basic open sets of the product topology, it suffices to take sets U_{α_i} which are *basic open* in X_{α_i}.

EXERCISE 5.3. Let $X = \prod_{\alpha\in I} X_\alpha$ be a product space. Prove the following assertions, establishing the product space is indeed a topological space:

(1) The set X is basic open.
(2) Every intersection of two basic open sets is basic open.
(3) The space X has the Hausdorff property.

DEFINITION 5.4. Let $\beta \in I$. The *projection function* on coordinate β is the function

$$\begin{aligned} \pi_\beta\colon \prod_{\alpha\in I} X_\alpha &\to X_\beta \\ (x_\alpha)_{\alpha\in I} &\mapsto x_\beta \end{aligned}$$

The product topology is the smallest topology for which all projections π_β (for $\beta \in I$) are continuous.

COROLLARY 5.5. *If a product space $\prod_{\alpha\in I} X_\alpha$ is compact, then for each $\alpha \in I$, the space X_α is compact.*

PROOF. Fix $\alpha \in I$. The space X_α a projection, and thus a continuous image, of the product space. Recall that a continuous image of a compact space is compact. $\square$

The converse of the last corollary is a central theorem in topology. Using ultrafilters, we obtain a short proof of this result.

LEMMA 5.6. *Let $\mathcal{B}$ be a basis for the topology of a topological space Y. A function $f\colon X \to Y$ is continuous if and only if for each $V \in \mathcal{B}$ in Y, the preimage $f^{-1}(V)$ is open in X.*

PROOF. Every open set in Y is a union of basic open sets in Y. $\square$

THEOREM 5.7 (Tychonoff). *Every product of compact spaces is compact: If X_α is a compact space for each $\alpha \in I$, then the product space $\prod_{\alpha\in I} X_\alpha$ is compact.*

PROOF. Remarkably, the proof given here is identical to that given in Theorem 4.12 for finite products.

Let $\mathcal{F}$ be an ultrafilter in $\prod_{\alpha\in I} X_\alpha$. For each $\alpha \in I$, the image $\pi_\alpha(\mathcal{F})$ of $\mathcal{F}$ is an ultrafilter in the compact space X_α. Let $x_\alpha \in X_\alpha$ be its limit point. We will show that $\mathcal{F}$ converges to $(x_\alpha)_{\alpha\in I}$.

Let U be an neighborhood of $(x_\alpha)_{\alpha\in I}$. Let

$$[\alpha_1, \dots, \alpha_n\,;\, U_{\alpha_1}, \dots, U_{\alpha_n}]$$

be a basic open set containing $(x_\alpha)_{\alpha\in I}$ and contained in U. Let $i \in \{1, \dots, n\}$. As $x_{\alpha_i} \in U_{\alpha_i}$, there is $A_{\alpha_i} \in \mathcal{F}$ with $\pi_{\alpha_i}(A_{\alpha_i}) \subseteq U_{\alpha_i}$. Then

$$A_{\alpha_1} \cap \dots \cap A_{\alpha_n} \in \mathcal{F},$$

and this is a subset of $[\alpha_1, \dots, \alpha_n\,;\, U_{\alpha_1}, \dots, U_{\alpha_n}]$, (and thus of U): For each element $y = (y_\alpha)_{\alpha\in I}$ in this set and every $i = 1, \dots, n$,

$$y_{\alpha_i} = \pi_{\alpha_i}(y) \in \pi_{\alpha_i}(A_{\alpha_i}) \subseteq U_{\alpha_i}. \qquad \square$$

As an application of Tychonoff's Theorem, we provide an alternative proof of the Full Compactness Theorem 2.1. This proof demonstrates that the notions of topological compactness and compactness for colorings are related in a natural manner.

We first observe that, for every set X, the family $P(X)$ of all subsets of X is a topology on X.

DEFINITION 5.8. A topological space is *discrete* if every set in this space is open. In other words, if its topology is $P(X)$.

A simple basis for the discrete topology on a set X is the family $\{\,\{x\} : x \in X\,\}$ of all single-element subsets of X. Since all subsets in a discrete topology are open, discrete topological spaces are not interesting in their own. But interesting topological spaces can be constructed from them, as we will see in the following proof.

A TYCHONOFF PRODUCT PROOF OF THE FULL COMPACTNESS THEOREM. Take the discrete topology on the set $\{1, \dots, k\}$. Being finite, this is a compact space. Consider the product space $Y = \prod_{x\in X}\{1, \dots, k\} = \{1, \dots, k\}^X$, in which all multiplied spaces are equal to the space $\{1, \dots, k\}$. By Tychonoff's Theorem, the space Y is compact.

The elements of Y are in one-to-one correspondence to the k-colorings $c\colon X \to \{1, \dots, k\}$ of X. Indeed, every such coloring c corresponds to the sequence $(c(x))_{x\in X}$.

Let $c = (c(x))_{x\in X} \in \{1, \dots, k\}^X$. The premise of the theorem asserts that there are $F \in \mathcal{A}$ and $i \in \{1, \dots, k\}$ such that $c(x) = i$ for all $x \in F$. Writing $F = \{x_1, \dots, x_n\}$, we have that

$$c \in [x_1, \dots, x_n\,;\, \{i\}, \dots, \{i\}]$$

We will denote this set by $[F\,;\, i]$. Then the family

$$\{\,[F\,;\, i] : F \in \mathcal{A},\ i = 1, \dots, k\,\}$$

is an open cover of Y. As Y is compact, this family has a finite subcover $\{[F_1\,;\, i_1], \dots, [F_n\,;\, i_n]\}$. Take $F = F_1 \cup \dots \cup F_n$. Let c be a k-coloring c of F. Extend it in an arbitrary manner to a k-coloring $\tilde{c}$ of the entire X. Then there is $m \leq n$ such that $\tilde{c} \in [F_m\,;\,i_m]$, and thus F_m is monochromatic for $\tilde{c}$. As c is define on F and agrees with $\tilde{c}$ there, the set F_m is monochromatic for c. □

6. Comments for Chapter 2

In Lemma 4.8 we provide a sufficient condition for a family of subsets of a set X to be a basis for a topology on X. To have a *necessary* and sufficient condition, one may replace item (2) of that corollary with the following one:

For all $U, V \in \mathcal{B}$ and each $x \in U \cap V$, there is $W \in \mathcal{B}$ with $x \in W \subseteq U \cap V$.

Zorn's Lemma (Lemma 1.12) is an important mathematical tool, used in many branches of mathematics to establish the existence of maximal objects, including for example bases for infinite-dimensional vector spaces and maximal ideals in rings. The proof of Zorn's Lemma uses, in a nontrivial manner, the Axiom of Choice: For each family $\{\, X_\alpha : \alpha \in I \,\}$ of nonempty sets, there is a *choice function*: a function f with domain I such that $f(\alpha) \in X_\alpha$ for all $\alpha \in I$. The Axiom of Choice does not assert that a choice function can be explicitly constructed or defined, but merely that it exists. This is what makes it a widely accepted axiom for the foundations of mathematics.

In the absence of the Axiom of Choice, many equivalences between the various principles used in our proofs can be proved (without appealing to results that necessitate the use of the Axiom of Choice). For example, Zorn's Lemma, the Axiom of Choice, and Tychonoff's Product Theorem 5.7 (for general, not necessarily Hausdorff, compact spaces) are equivalent. The Ultrafilter Theorem (Lemma 1.13) is deductively weaker than the Axiom of Choice, but each of the following assertions is equivalent to the Ultrafilter Theorem: Tychonoff's Product Theorem 5.7 (for compact Hausdorff spaces); Every power $\{1, \dots, k\}^X$ is compact (used in the proof of the Full Compactness Theorem given in Section 5); Theorem 4.6 (characterizing compactness by convergence of ultrafilters); and the Full Compactness Theorem. A thorough treatment of results of this type is available in Horst Herrlich's book, *Axiom of Choice*, Springer, 2006.

CHAPTER 3

The Stone–Čech compactification

1. The space of ultrafilters

For brevity, let us fix an infinite set X throughout this section.

Definition 1.1. βX is the set of all ultrafilters on X.

We would like to consider the set βX as a topological space, and its elements as points in that space. Thus, it would be natural to henceforth denote points βX (which happen to be ultrafilters on X) by lowercase letters such as p, q, etc.

Definition 1.2. For a set $A \subseteq X$, let $[A] := \{ p \in \beta X : A \in p \}$.

Exercise 1.3. Prove that the function $A \mapsto [A]$, defined on $P(X)$, has the following properties:

(1) $[\emptyset] = \emptyset$ and $[X] = \beta X$.
(2) For all $A, B \subseteq X$:
   (a) $[A] \subseteq [B]$ if and only if $A \subseteq B$.
   (b) $[A] = [B]$ if and only if $A = B$.
   (c) $[A] \cup [B] = [A \cup B]$;
   (d) $[A] \cap [B] = [A \cap B]$;
   (e) $[A^{\mathsf{c}}] = [A]^{\mathsf{c}}$.

Exercise 1.4. Consider the case $X = \mathbb{N}$.

(1) Find sets $A_1, A_2, \ldots \subseteq \mathbb{N}$ such that $[\bigcup_{n=1}^{\infty} A_n] \neq \bigcup_{n=1}^{\infty} [A_n]$.
(2) Find sets $A_1, A_2, \ldots \subseteq \mathbb{N}$ such that $[\bigcap_{n=1}^{\infty} A_n] \neq \bigcap_{n=1}^{\infty} [A_n]$.

Hint: (1) implies (2). Consider one-element sets.

By Exercise 1.3, the family $\mathcal{B} = \{ [A] : A \subseteq X \}$ satisfies the conditions of Lemma 4.8 for being a basis for a topology on βX.

Definition 1.5. The topology of βX is the one with basic open sets $[A]$ (for $A \subseteq X$).

Since $[A]^{\mathsf{c}} = [A^{\mathsf{c}}]$ for all $A \subseteq X$, the sets $[A]$ are *clopen*, that is, simultaneously closed and open.

Theorem 1.6. *The topological space βX is compact.*

Proof. Assume otherwise, and consider an open cover of βX with no finite subcover. We may assume that this cover is of the form $\{ [A_\alpha] : \alpha \in I \}$, for some index set I. For all $\alpha_1, \ldots, \alpha_n \in I$, since

$$[A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}] = [A_{\alpha_1}] \cup \cdots \cup [A_{\alpha_n}] \neq \beta X = [X],$$

we have that $A_{\alpha_1} \cup \cdots \cup A_{\alpha_n} \neq X$, or, equivalently, that $A_{\alpha_1}^{\mathsf{c}} \cap \cdots \cap A_{\alpha_n}^{\mathsf{c}} \neq \emptyset$. It follows that the family $\{ A_\alpha^{\mathsf{c}} : \alpha \in I \}$ extends to an ultrafilter $p \in \beta X$. Let $\alpha \in I$ be such that $p \in [A_\alpha]$. Then $A_\alpha, A_\alpha^{\mathsf{c}} \in p$; a contradiction. □

Definition 1.7. For $x \in X$, let $p_x \in \beta X$ be the principal ultrafilter determined by x.

The function from X to βX defined by $x \mapsto p_x$ is bijective. We identify each principal ultrafilter p_x with the point x. Under this identification, we have that $X \subseteq \beta X$, and the set X becomes a topological subspace of βX. We will see that the topology induced on X is the simplest possible.

Recall that a topological space X is *discrete* if all subsets of X are open. A point $x \in X$ is *isolated* if the set $\{x\}$ is open. In other words, if there is a neighborhood of x containing no other point. A topological space is discrete if and only if all of its points are isolated.

A subset of a topological space is *dense* if its closure is the entire space. One may interpret the following lemma as asserting that every dictatorship is isolated, but in every neighborhood of any government one may find a dictatorship.

LEMMA 1.8. *Every point of X is isolated in βX, but the set X is dense in βX.*

Formally: For each $x \in X$, the set $\{p_x\}$ is open in βX, and the closure of the set $\{\, p_x : x \in X \,\}$ is βX.

PROOF. Let $x \in X$. By definition, $p \in [\{x\}]$ if and only if $\{x\} \in p$, and the latter property is equivalent to $p = p_x$. Thus, $\{p_x\} = [\{x\}]$, a basic open set.

Let $q \in \beta X$. For each basic neighborhood $[A]$ of q, we have that $A \in q$ and, therefore, $A \neq \emptyset$. Fix $x \in A$. Then $A \in p_x$, that is, $p_x \in [A]$. $\square$

EXERCISE 1.9. Prove the following generalization of Lemma 1.8: For each $A \subseteq X$, we have that $\overline{A} = [A]$. (Formally: $\overline{\{\, p_x : x \in A \,\}} = [A]$).

We say that a topological space X is a subspace of another space Y if $X \subseteq Y$ and the induced topology on X coincides with the original one.

DEFINITION 1.10. A *compactification* of a topological space X is a compact space K such that X is a dense subspace of K.

If we think of a set X with no prescribed topology as a discrete topological space, then Lemma 1.8 implies that the space βX is a compactification of X. The space βX is called the *Stone–Čech compactification* of X.

2. Excursion: The origin of the open sets in βX

We will see here that the topology we have defined on βX is the natural one. Let I be a set, and consider the set $\{0,1\}$ as a discrete topological space. By Tychonoff's Product Theorem (Theorem 5.7), the space $\{0,1\}^I = \prod_{\alpha\in I}\{0,1\}$ is compact. The basic open sets in this space are

$$[\alpha_1,\dots,\alpha_n\,;\,\{i_1\},\dots,\{i_n\}] = \left\{\, (x_\alpha)_{\alpha\in I} \in \{0,1\}^I : x_{\alpha_1} = i_1,\dots,x_{\alpha_n} = i_n \,\right\},$$

where n is a natural number, $\alpha_1,\dots,\alpha_n \in I$, and $i_1,\dots,i_n \in \{0,1\}$. By reordering the elements, we may assume that there is $l \leq n$ such that $i_1 = \dots = i_l = 0$ and $i_{l+1} = \dots = i_n = 1$. This way, the basic open sets may be denoted as

$$[\alpha_1,\dots,\alpha_l\,;\,\beta_1,\dots,\beta_m] := \left\{\, (x_\alpha)_{\alpha\in I} \in \{0,1\}^I : x_{\alpha_1} = \dots = x_{\alpha_l} = 0, x_{\beta_1} = \dots = x_{\beta_m} = 1 \,\right\}.$$

We may identify the set $\{0,1\}^I$ with the family $P(I)$ of all subsets of I, by identifying each set $J \subseteq I$ with its *characteristic sequence* $(x_\alpha)_{\alpha\in I} \in \{0,1\}^I$, define by

$$x_\alpha = \begin{cases} 1 & \alpha \in J \\ 0 & \alpha \notin J \end{cases}$$

for all $\alpha \in I$. This identification transports the Tychonoff product topology on $\{0,1\}^I$ into $P(I)$. The basic open sets in $P(I)$ are those of the form

$$[\alpha_1, \dots, \alpha_l\,;\, \beta_1, \dots, \beta_m] = \{\, J \in P(I) : \alpha_1, \dots, \alpha_l \notin J, \beta_1, \dots, \beta_m \in J \,\}\,.$$

This is the *product topology on* $P(I)$.

The set I could be any set. In particular, we can take $I = P(X)$ for a prescribed set X. Then the set βX of all ultrafilters of X is a subset of $P(I)$. The following proposition asserts, in effect, that the topology on βX is the one you get from the product topology.

Proposition 2.1. *Let X be a set and $I = P(X)$. Equip the set $P(I)$ with the product topology. Then the topology of βX coincides with the subspace topology induced by the space $P(I)$.*

Proof. Consider the induced topology on βX. The basic open sets are the intersections of basic open sets in $P(I)$ with βX. By the above discussion, a basic open set in $P(I)$ is determined by elements $A_1, \dots, A_l, B_1, \dots, B_m \in I = P(X)$, and the basic open set is

$$[A_1, \dots, A_l\,;\, B_1, \dots, B_m] = \{\, p \in \beta X : A_1, \dots, A_l \notin p,\ B_1, \dots, B_m \in p \,\}\,.$$

Let $p \in \beta X$. As p is an ultrafilter, we have that the following assertions are equivalent:

(1) $A_1, \dots, A_m \notin p$, $B_1, \dots, B_r \in p$;
(2) $A_1^{\mathsf{c}}, \dots, A_l^{\mathsf{c}}, B_1, \dots, B_m \in p$;
(3) $A_1^{\mathsf{c}} \cap \cdots \cap A_l^{\mathsf{c}} \cap B_1 \cap \cdots \cap B_m \in p$.

Thus, taking $A := A_1^{\mathsf{c}} \cap \cdots \cap A_l^{\mathsf{c}} \cap B_1 \cap \cdots \cap B_m$, we see that the basic open set is

$$[A] := \{\, p \in \beta X : A \in p \,\}\,.$$

These are exactly the basic open sets in the original topology of βX. □

With this understanding, compactness of βX follows naturally.

Theorem 2.2. *The space βX is compact.*

Proof. Let $I = P(X)$, and consider the space $P(I)$ with the product topology. Since $P(I)$ is a compact space, it suffices to observe that βX is a closed subset of $P(I)$.

Let $q \in P(I) \setminus \beta X$. We will show that q has a neighborhood disjoint of βX. Indeed, every possible reason why q is not an ultrafilter defines such an open set. This is so since all reasons are in terms of membership or non-membership of certain sets to q. For example:

(1) If $X \notin q$ then $q \in [X\,;\,]$, which is disjoint of βX.
(2) If $\emptyset \in q$ then $q \in [\,;\,\emptyset]$, which is disjoint of βX.
(3) If there is a set $B \subseteq X$ such that $B \supseteq A \in q$ and $B \notin q$, then $q \in [B\,;\,A]$, which is disjoint of βX. □

Exercise 2.3. Complete the consideration of all cases in the proof of Theorem 2.2.

3. The Extension Theorem

Lemma 3.1 (Regularity). *Let K be a compact space and $x \in K$. For each neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.*

Proof. As the space K is compact, its closed subset U^{c} is compact, too. For each $y \in U^{\mathsf{c}}$, chose disjoint open neighborhoods V_y and U_y of x and y, respectively. Then $U^{\mathsf{c}} \subseteq \bigcup_{y \in U^{\mathsf{c}}} U_y$. As U^{c} is compact, there are $y_1, \dots, y_n \in U^{\mathsf{c}}$ such that $U^{\mathsf{c}} \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$. Let

$$\begin{aligned} C &= U_{y_1}^{\mathsf{c}} \cap \cdots \cap U_{y_n}^{\mathsf{c}}; \\ V &= V_{y_1} \cap \cdots \cap V_{y_n}. \end{aligned}$$

Then V is a neighborhood of x and $V \subseteq C$. As the set C is closed, we have that $\overline{V} \subseteq C$. $\square$

EXERCISE 3.2. Let $\mathcal{F}$ be a filter in a topological space that converges to a point x. Prove that for each $A \in \mathcal{F}$ we have that $x \in \overline{A}$.

THEOREM 3.3 (Extension). *Let K be a compact space. Every function $f\colon X \to K$ extends, in a unique manner, to a continuous function $\bar{f}\colon \beta X \to K$.*

$$\begin{tikzcd} \beta X \arrow[dr, dotted, "\bar{f}"] & \\ X \arrow[u, hook, "\mathrm{id}"] \arrow[r, "f"] & K \end{tikzcd}$$

PROOF. Uniqueness follows from the density of X in βX. We prove existence.

For each $p \in \beta X$, $f(p)$ is an ultrafilter in the compact space K, and thus converges to a unique point in K. Define

$$\bar{f}(p) := \lim f(p).$$

The function $\bar{f}$ extends f: For each $x \in X$, we have that

$$f(p_x) = \{\, f(A) : A \in p_x \,\},$$

and the set $\{f(x)\}$ is in the latter set. Thus, $\bar{f}(x) = \lim f(p_x) = f(x)$.

The function $\bar{f}$ is continuous: Let $p \in \beta X$. Let V be a neighborhood of $\bar{f}(p)$ in K. As $\lim f(p) = \bar{f}(p) \in V$, there is $f(A) \in f(p)$ (with $A \in p$) such that $f(A) \subseteq V$. The set $[A]$ is a neighborhood of p, and for $q \in [A]$ we have that $A \in q$, and thus $f(A) \in f(q)$. As V is closed, we have that $\bar{f}(q) = \lim f(q) \in \overline{f(A)} \subseteq \overline{V}$. This is almost what we need.

By the Regularity Lemma, there is a neighborhood W of $\bar{f}(p)$ such that $\overline{W} \subseteq V$. Carrying out the preceding argument with W instead of V, we have that $\bar{f}(q) \in \overline{W} \subseteq V$ for all $q \in [A]$. $\square$

EXERCISE 3.4. Prove, using the Hausdorff property, the uniqueness of the extension $\bar{f}$ in the Extension Theorem.

A tip for the next exercise: By the characterization of ultrafilters as maximal filters, in order to establish equality of an ultrafilter p to a set q, it suffices to prove that q is a filter and $p \subseteq q$.

EXERCISE 3.5. Let $f\colon X \to X$. As $X \subseteq \beta X$, we have in particular that $f\colon X \to \beta X$. Let $\bar{f}\colon \beta X \to \beta X$ be the unique continuous extension of f. Prove that, in this case,

$$\bar{f}(p) = f(p)^{\uparrow} := \{\, B \subseteq X : \exists A \in p,\ f(A) \subseteq B \,\}$$

for all $p \in \beta X$.

EXERCISE 3.6. Let $f\colon X \to \beta X$, and let $\bar{f}\colon \beta X \to \beta X$ be its unique continuous extension. Prove that, in this case,

$$\bar{f}(p) = \{\, B \subseteq X : \exists A \in p \forall x \in A,\ B \in f(x) \,\}$$

for all $p \in \beta X$.

4. Multiplication in βS

DEFINITION 4.1. A *semigroup* is a nonempty set S equipped with an associative binary operator $*$. Explicitly:

(1) To each pair of elements $a, b \in S$, a unique element $a * b \in S$ is assigned.
(2) For all $a, b, c \in S$, we have that $(a * b) * c = a * (b * c)$.

The multiplication symbols is usually omitted, writing ab instead of $a * b$.

Let S be a semigroup. We extend the multiplication operator from $S \times S$ to $\beta S \times \beta S$ by applying the Extension Theorem theorem twice: First, we extend it from $S \times S$ to $S \times \beta S$ by fixing the left coordinate and using the extension theorem on the right coordinate, and then we extend it from $S \times \beta S$ to $\beta S \times \beta S$ by fixing the right coordinate and using the extension theorem on the left coordinate. The exact details are provided in the following definition.

Definition 4.2. Let S be a semigroup.

(1) Fix an element $a \in S$. Let $L_a \colon S \to S \subseteq \beta S$ be the function of left multiplication by a, that is,

$$L_a(x) := ax$$

for all $x \in X$. By the Extension Theorem, the function L_a extends uniquely to a continuous function $\bar{L}_a \colon \beta S \to \beta S$.

$$\begin{tikzcd} \beta S \arrow[dr, dotted, "\bar{L}_a"] & \\ S \arrow[u, hook, "\mathrm{id}"] \arrow[r, "L_a"] & \beta S \end{tikzcd}$$

Define $aq := \bar{L}_a(q)$ for all $q \in \beta S$. Keep in mind that this function is continuous in its argument q.

(2) Fix an element $q \in \beta S$. Let $R_q \colon S \to \beta S$ be the function of right multiplication by q, that is,

$$R_q(x) := xq$$

for all $x \in X$. By the Extension Theorem, the function R_q extends uniquely to a continuous function $\bar{R}_q \colon \beta S \to \beta S$.

$$\begin{tikzcd} \beta S \arrow[dr, dotted, "\bar{R}_q"] & \\ S \arrow[u, hook, "\mathrm{id}"] \arrow[r, "R_q"] & \beta S \end{tikzcd}$$

Define $pq := \bar{R}_q(p)$ for all $p \in \beta S$. This function is continuous in its argument p.

Definition 4.2 defines the product pq for all $p, q \in \beta S$, in a way that the function

$$\begin{array}{rcl} \beta S \times \beta S & \to & \beta S \\ (p, q) & \mapsto & pq \end{array}$$

is continuous in its first argument p, and its restriction

$$\begin{array}{rcl} S \times \beta S & \to & \beta S \\ (a, q) & \mapsto & aq \end{array}$$

to $S \times \beta S$ is continuous in its second argument q.

Exercise 4.3. Prove that every composition of continuous functions is continuous.

Lemma 4.4. *The set βS, with the binary operator $(p, q) \mapsto pq$, is a semigroup.*

Proof. Let $p, q, r \in \beta S$. We need to prove that $(pq)r = p(qr)$.
1. For all $x, y, z \in S$, we have that

$$\bar{L}_{xy}(z) = L_{xy}(z) = (xy)z = x(yz) = L_x \circ L_y(z) = \bar{L}_x \circ \bar{L}_y(z),$$

that is, the continuous functions $\bar{L}_{xy}$ and $\bar{L}_x \circ \bar{L}_y$ coincide on S. As S is dense in βS, these two functions coincide on all of βS. In particular, we have that

$$(xy)r = \bar{L}_{xy}(r) = \bar{L}_x \circ \bar{L}_y(r) = x(yr).$$

2. By the last equation, we have that $\bar{R}_r \circ \bar{L}_x(y) = \bar{L}_x \circ \bar{R}_r(y)$ for all $y \in S$, that is, the continuous functions $\bar{R}_r \circ \bar{L}_x$ and $\bar{L}_x \circ \bar{R}_r$ coincide on S. As S is dense in βS, these two functions coincide on all of βS. In particular, we have that

$$(xq)r = \bar{R}_r \circ \bar{L}_x(q) = \bar{L}_x \circ \bar{R}_r(q) = x(qr).$$

3. By the last equation, the continuous functions $\bar{R}_r \circ \bar{R}_q$ and $\bar{R}_{qr}$ coincide on S, and thus on βS. In particular, we have that $(pq)r = p(qr)$. □

A *right topological semigroup* is a semigroup with a topology such that, for each constant $c \in S$, multiplication by c on the right, $x \mapsto xc$, is continuous. Following the tradition of naming algebraic structures as groups, rings, bands, etc., we introduce the following name for an algebraic structure.

Definition 4.5. A *company* is a *compact* right topological semigroup.

Corollary 4.6. *For each semigroup S, the semigroup βS is a company.* □

The following combinatorial characterization of multiplication in βS will help shortening many later arguments. This characterization is due to Glazer. Intuitively, it asserts that $A \in pq$ if and only if there are p-many elements b for which there are q-many elements c with $bc \in A$.

Theorem 4.7 (Product Characterization). *let S be a semigroup and $p, q \in \beta S$. Then:*

(1) $pq = \{\, A \subseteq S : \exists B \in p \forall b \in B \exists C \in q,\ bC \subseteq A \,\}$. *In other words, a set A is in pq if and only if there is $B \in p$ such that, for each $b \in B$, $bC \subseteq A$ for some $C \in q$.*
(2) *For each $A \in pq$ there are $b \in S$ and $C \in q$ such that $bC \subseteq A$.*

Proof. (2) follows from (1). We prove (1). Note that $[A] \cap S = A$ (in particular, $A \subseteq [A]$) and $\overline{A} = [A]$ for all sets $A \subseteq S$.

($\Rightarrow$) $pq \in [A]$. By continuity of right multiplication by q ($\bar{R}_q$), there is a neighborhood $[B]$ of p ($B \in p$) such that $Bq \subseteq [B]q \subseteq [A]$.

Let $b \in B$. Then $bq \in Bq \subseteq [A]$. By continuity of left multiplication by an element of S ($\bar{L}_b$), there is a neighborhood $[C]$ of q ($C \in q$) such that $bC \subseteq b[C] \subseteq [A]$. Thus, $bC \subseteq [A] \cap S = A$.

($\Leftarrow$) For each $b \in B$, let $C \in q$ be such that $bC \subseteq A$. By continuity of left multiplication by b, we have that $b\overline{C} \subseteq \overline{bC} \subseteq \overline{A}$. Since $q \in [C] = \overline{C}$, we have that $bq \in \overline{A}$. In summary, $Bq \subseteq \overline{A}$. By continuity of right multiplication by q, we have that $\overline{B}q \subseteq \overline{Bq} \subseteq \overline{A}$. Since $p \in [B] = \overline{B}$, $pq \in \overline{A} = [A]$, that is, $A \in pq$. □

A diagram containing some of the information in Theorem 4.7 may help remembering it:

$$\begin{array}{cccc} A \in & p & \cdot & q \\ & | & & | \\ & B & & | \\ & | & & | \\ A \supseteq & b & \cdot & C \end{array}$$

According to our identification of the elements of S with the principal ultrafilters in βS, we have that S is a subsemigroup of βS.

Exercise 4.8. Let S be a semigroup. Prove, using the Product Characterization Theorem, that for all $a, b \in S$ we have that $p_a p_b = p_{ab}$.

5. When is $\beta S \setminus S$ a company?

As the set $\{s\}$ is open in βS for all $s \in S$, the set S is open in βS. Thus, the set $\beta S \setminus S$ is topologically closed in βS. If, in addition, the set $\beta S \setminus S$ is closed under multiplication, then it is a company, and there is an idempotent in $\beta S \setminus S$. We describe this sufficient condition in terms of the semigroup S.

Definition 5.1. A semigroup S is *moving* if S is infinite, and for all finite $F \subseteq S$ and infinite $A \subseteq S$, there are $a_1, \dots, a_k \in A$ such that, for all but finitely many $s \in S$,

$$\{a_1 s, \dots, a_k s\} \nsubseteq F.$$

A *group* is a semigroup with an element e such that $se = es = s$ for all s, and for each s there is t such that $st = ts = e$. A semigroup S is *left cancellative* if, for all $a, b, c \in S$, if $ca = cb$ then $a = b$. *Right cancellative* semigroups are defined similarly. A function $f \colon X \to Y$ is *finite-to-one* if for all $y \in Y$, the set of preimages $f^{-1}(y) = \{\, x \in X : f(x) = y \,\}$ of y is finite.

Exercise 5.2. Let S be a semigroup. If S is a group, then it is right cancellative and left cancellative. If S is right cancellative or left cancellative, then it is moving. If left multiplication in S is finite-to-one (i.e., the functions $L_a \colon x \mapsto ax$ are finite-to-one), then S is moving.

Theorem 5.3. *Let S be a semigroup. The following assertions are equivalent:*

1. *$\beta S \setminus S$ is a subsemigroup of βS.*
2. *S is moving.*

Proof. (2) $\Rightarrow$ (1): Let $p, q \in \beta S \setminus S$ and assume that $pq = s \in S$. Then $\{s\} \in pq$. By the Product Characterization Theorem, there is $A \in p$ such that for each $a \in A$ there is $B_a \in q$ such that $aB_a \subseteq \{s\}$. Let $a_1, \dots, a_k \in A$. Then the set $B := B_{a_1} \cap \dots \cap B_{a_k}$ is in q, and thus infinite, and

$$\{a_1, \dots, a_k\}B \subseteq \{s\}.$$

Thus, S is not moving.

(1) $\Rightarrow$ (2): Let $F \subseteq S$ be finite and $A \subseteq S$ be infinite, such that for all $a_1, \dots, a_k \in A$, the set $\{\, s \in S : a_1 s, \dots, a_k s \in F \,\}$ is infinite. For each $a \in A$, let

$$B_a = \{\, s \in S : as \in F \,\}.$$

Then, for all $a_1, \dots, a_k \in A$, the intersection $B_{a_1} \cap \dots \cap B_{a_k}$ is infinite. Let $q \in \beta S \setminus S$ be an ultrafilter containing all sets B_a for $a \in A$. Since the set A is infinite, there is an ultrafilter $p \in \beta S \setminus S$ with $A \in p$. We will show that $F \in pq$, and therefore $pq \in S$. Indeed, $A \in p$ and for all $a \in A$, the set B_a is in q, and $aB_a \subseteq F$. By the Product Characterization Theorem, we have that $F \in pq$. $\square$

6. Idempotents

Lemma 6.1 (Finite Intersection Property). *Let K be a compact space, and let $\{\, C_\alpha : \alpha \in I \,\}$ be a family of closed sets in K such that every intersection of finitely many members of this family is nonempty. Then the entire intersection, $\bigcap_{\alpha \in I} C_\alpha$, is nonempty.*

Proof. If $\bigcap_{\alpha \in I} C_\alpha = \emptyset$, then the open cover $\bigcup_{\alpha \in I} C_\alpha^{\mathsf{c}} = K$ has a finite subcover $C_{\alpha_1}^{\mathsf{c}} \cup \dots \cup C_{\alpha_n}^{\mathsf{c}} = K$. Then $C_{\alpha_1} \cap \dots \cap C_{\alpha_n} = \emptyset$; a contradiction. $\square$

Lemma 6.2. *Let K be a compact space. A set $C \subseteq K$ is closed if and only if C is compact.*

Proof. We already know that closed subsets of compact spaces are compact. The proof of the converse implication is similar to the proof of the Regularity Lemma (Lemma 3.1). $\square$

EXERCISE 6.3. Complete the proof of Lemma 6.2.

Thus, when working in compact spaces, we will freely switch among the terms "closed" and "compact".

EXERCISE 6.4. Prove that for each point x in a topological space X, the one-point set $\{x\}$ is closed.

An *idempotent* in a semigroup S is an element $e \in S$ satisfying $e^2 := ee = e$.

THEOREM 6.5. *Every company S has an idempotent element.*

PROOF. We will use the following variation of Zorn's Lemma (Lemma 1.12): Let $\mathcal{A}$ be a nonempty family of sets, with the property that for each chain $\{ A_\alpha : \alpha \in I \}$ in $\mathcal{A}$, we have that $\bigcap_{\alpha\in I} A_\alpha \in \mathcal{A}$. Then there is a minimal element $A \in \mathcal{A}$. (I.e., such that there is no $B \in \mathcal{A}$ with $B \subsetneq A$.)

A subcompany of S is a subset that is a company with respect to the (induced) multiplication and topology of S. We will apply Zorn's Lemma to find a minimal subcompany of S, and show that this subcompany must be of the form $\{e\}$. (In particular, $ee = e$.)

The family of all subcompanies of S satisfies the conditions of Zorn's Lemma: S is there, and the intersection of a chain of subcompanies is a company, by the finite intersection property of compact sets. By Zorn's Lemma, there is a minimal subcompany $T \subseteq S$.

Fix an element $e \in T$. As right multiplication by e is continuous and the set T is compact, the set $Te = \{ te : t \in T \}$ is also compact. As $e \in T$, we have that $Te \subseteq T$ and Te is closed under multiplication. Thus, the set Te is a company. By the minimality of T, we have that $Te = T$, and therefore $e \in Te$. It follows that the *stabilizer* of e in T, defined as

$$\mathrm{stab}_T(e) := \{ t \in T : te = e \} = R_e^{-1}(\{e\}) \cap T,$$

is nonempty. The stabilizer is a subsemigroup of T: For t_1, t_2 in the stabilizer, we have that $(t_1t_2)e = t_1(t_2e) = t_1e = e$. As the set $\{e\}$ is closed and the function R_e is continuous, the set $R_e^{-1}[\{e\}]$ is closed, and therefore so is its intersection with T. Thus, the set $\mathrm{stab}_T(e)$ is compact, and is therefore a subcompany of T. By the minimality of T, we have that $\mathrm{stab}_T(e) = T$ and, in particular, that $e \in \mathrm{stab}_T(e)$, that is, $ee = e$. This establishes the theorem, but it also follows that $\{e\}$ is a company contained in T, and therefore $T = \{e\}$. ☐

COROLLARY 6.6. *Let S be a semigroup. In each closed subsemigroup of βS there is an idempotent element.* ☐

We will usually consider $\mathbb{N}$ as a semigroup with respect to its additive structure. Thus, an idempotent element of $\beta\mathbb{N}$ is an element $e \in \mathbb{N}$ with $e + e = e$.

EXERCISE 6.7. Prove that for every idempotent $e \in \beta\mathbb{N}$ and each n, we have that $n\mathbb{N} \in e$. *Hint*: Let $\varphi\colon \mathbb{N} \to \mathbb{Z}_n$ be the canonical homomorphism $\varphi(k) = k \mod n$. The finite semigroup $\mathbb{Z}_n$, with the discrete topology, is compact. Thus, φ extends to a continuous homomorphism (!) $\bar{\varphi}\colon \beta\mathbb{N} \to \mathbb{Z}_n$. Then $\bar{\varphi}(e)$ is an idempotent in $\mathbb{Z}_n$, and is therefore equal to 0. Apply the continuity of $\bar{\varphi}$.

THEOREM 6.8 (Idempotent Characterization). *Let S be a semigroup and let $e \in \beta S$ be an idempotent. For each $A \in e$, there is $a \in A$ such that:*

There is a set $A' \subseteq A$ in e such that $aA' \subseteq A$.

Moreover:

1. *The set of elements a with the quoted property is in e.*
2. *If e is nonprincipal, then we may request that $a \notin A'$.*

(3) *Item (1) above characterizes the property "e is an idempotent in βS".*

PROOF. Let $A \in e = e^2$. By the Product Characterization Theorem, there is $B \in e$ such that, for each $a \in B$, there is $C \in e$ with $aC \subseteq A$. Fix $a \in B \cap A$, and take the corresponding set C. Then the set $A' = C \cap A$ is in e, and $aA' \subseteq aC \subseteq A$. This establishes (1) and, in particular, the quoted assertion.

(2) Since e is nonprincipal, the set $\{a\}$ is not in e and $\{a\}^{\mathsf{c}} \in e$. Take $A' = C \cap A \cap \{a\}^{\mathsf{c}} = C \cap A \setminus \{a\}$.

(3) By the Product Characterization Theorem and (1), we have that $e \subseteq e^2$. As e and e^2 are ultrafilters, equality holds. $\square$

It follows directly from the definition that every subsemigroup of a moving semigroup is moving.

THEOREM 6.9. *Let S be a semigroup. If S has a moving subsemigroup, then there is an idempotent $e \in \beta S \setminus S$.*

PROOF. Let T be a moving subsemigroup of S. By Theorem 5.3, there is an idempotent $e \in \beta T \setminus T$. Let

$$e^{\uparrow} = \{\, A \subseteq S : \exists B \in e,\, A \supseteq B \,\},$$

the closure of e under taking supersets. Then $e^{\uparrow} \in \beta S$. By the Idempotent Characterization Theorem, $e^{\uparrow}$ is an idempotent in βS. As all elements of $e^{\uparrow}$ are infinite (given that $e \in \beta T \setminus T$), we have that $e^{\uparrow} \in \beta S \setminus S$. $\square$

EXERCISE 6.10. Prove that if there are no idempotents in a semigroup S, then S has a subsemigroup isomorphic to $\mathbb{N}$. In particular, in this case S has a moving subsemigroup.

7. Comments for Chapter 3

The Stone–Čech compactification occurs implicitly in Andrey N. Tychonoff's paper *Über die topologische Erweiterung von Räumen*, Mathematische Annalen, 1930. Explicit introductions of this space were given by Marshall Stone, in his paper *Applications of the theory of Boolean rings to general topology*, Transactions of the American Mathematical Society, 1937, and by Eduard Čech, in his paper *On bicompact spaces*, Annals of Mathematics, 1937.

We have described the basic open sets in βX. This determines the open sets of βX as unions of basic open sets. We provide here an explicit description of the open sets in βX.

Let $U = \bigcup_{\alpha \in I}[A_\alpha]$ be a union of basic open sets. Let $p \in \beta X$ be such that $p \notin U$. Then $A_\alpha \notin p$, and thus $B_\alpha := A_\alpha^{\mathsf{c}} \in p$, for all $\alpha \in I$. The sets B_α generate a filter $\mathcal{F}$ contained in p. The converse implications also hold, and we have that $p \notin U$ if and only if $\mathcal{F} \subseteq p$. Thus, the open sets in βX are the sets of the form

$$[\mathcal{F}] = \{\, p \in \beta X : \mathcal{F} \nsubseteq p \,\},$$

for $\mathcal{F}$ a filter on X. Equivalently, *closed* sets in βX are those of the form $\{\, p \in \beta X : \mathcal{F} \subseteq p \,\}$.

Theorem 5.3 is due to Hindman, *The ideal structure of the space of κ-uniform ultrafilters on a discrete semigroup*, Rocky Mountain Journal of Mathematics, 1986 (Theorem 2.5 with $\kappa = \omega$).

De Gruyter Expositions in Mathematics, Theorem 4.28. We did not include in Exercise 5.2 semigroups with finite-to-one *right* multiplication. The reason is that there are such semigroups that are not moving (Benjamin Steinberg, Answer to *Mathoverflow* question 164050, 2014). Fortunately, for our purposes this is not crucial: A semigroup S is *periodic* if every elements of S generates a finite semigroup. A semigroup S is *right (left) zero* if $ab = b$ ($ab = a$) for all $a, b \in S$. Lev N. Shevrin (*On the theory of periodic semigroups*, Izvestija Vysših Učebnyh

Zavedeniĭ Matematika, 1974) proved that every infinite semigroup has a subsemigroup of one of the following types:

(1) $(\mathbb{N}, +)$.
(2) An infinite periodic group.
(3) An infinite right zero or left zero semigroup.
(4) $(\mathbb{N}, \vee)$, where $m \vee n := \max\{m, n\}$.
(5) $(\mathbb{N}, \wedge)$, where $m \wedge n := \min\{m, n\}$.
(6) An infinite semigroup S with S^2 finite.
(7) The fan semilattice $(\mathbb{N}, \wedge)$, with $m \wedge n = 1$ for distinct m, n (and $n \wedge n = n$ for all n).

Assume that right multiplication in S is not finite-to-one. Then S does not have a subsemigroup of the type (6) or (7). Thus, it must have a subsemigroup of one of the remaining types (1)–(5), which are all moving (!). Thus, Theorem 6.9 applies to semigroups with finite-to-one right multiplication as well.

CHAPTER 4

Monochromatic finite sums and products

1. Hindman's Theorem

DEFINITION 1.1. Let S be a semigroup, and let $a_1, a_2, \dots \in S$. $\mathrm{FP}(a_1, a_2, \dots)$ is the set of all finite products $a_{i_1}a_{i_2}\cdots a_{i_n}$ with $i_1 < i_2 < \cdots < i_n$, for arbitrary n.

In particular, we have that $a_1, a_2, \dots \in \mathrm{FP}(a_1, a_2, \dots)$. When considering the semigroup $\mathbb{N}$ with respect to addition, we write $\mathrm{FS}(a_1, a_2, \dots)$ instead of $\mathrm{FP}(a_1, a_2, \dots)$, since this is a set of *finite sums*.

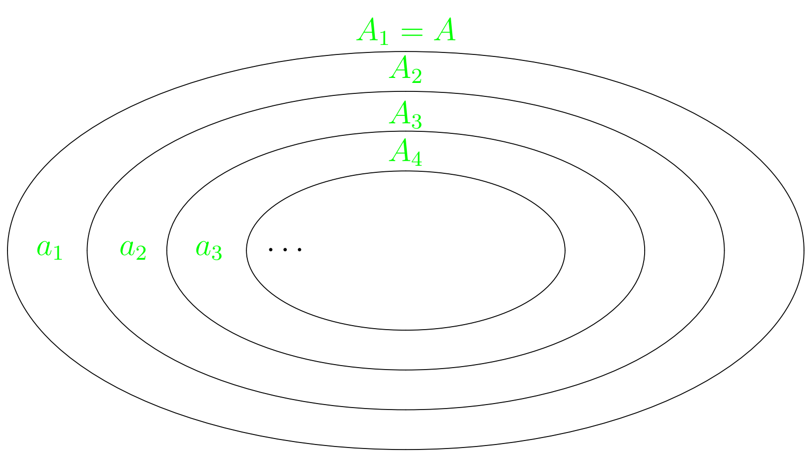
Schur's Coloring Theorem (Theorem 3.3) may be restated as follows: for each finite coloring of $\mathbb{N}$, there are natural numbers x and y such that the numbers x, y and $x + y$ have the same color. The following theorem, due to Neil Hindman, is much stronger.

THEOREM 1.2 (Hindman). *For each finite coloring of* $\mathbb{N}$*, there are distinct natural numbers* $a_1, a_2, \dots$ *such that the set* $\mathrm{FS}(a_1, a_2, \dots)$ *is monochromatic.*

Consider the semigroup $(\mathbb{N}, +)$. As its extension $(\beta\mathbb{N}, +)$ is a company, there is an idempotent $e \in \beta\mathbb{N}$, that is, such that $e + e = e$. As there are no idempotents in $(\mathbb{N}, +)$, we have that $e \in \beta\mathbb{N} \setminus \mathbb{N}$. Thus, Hindman's Theorem follows from the following theorem. Moreover, by Theorem 6.9, every semigroup that contains a moving subsemigroup satisfies the premise of the following theorem.

THEOREM 1.3 (Galvin–Glazer). *Let* S *be an infinite semigroup. If there is an idempotent* $e \in \beta S \setminus S$*, then for each finite coloring of* S *there are distinct elements* $a_1, a_2, \dots \in S$ *such that the set* $\mathrm{FP}(a_1, a_2, \dots)$ *is monochromatic.*

PROOF. Fix an idempotent $e \in \beta S \setminus S$. Let a finite coloring of S be given. As e is an ultrafilter on S, there is in e a monochromatic set A. We proceed as in the "good case" in the proof of Ramsey's Theorem:



By the Idempotent Characterization Theorem (Theorem 6.8), there are an element $a_1 \in A$ and a set $A_2 \subseteq A \setminus \{a_1\}$ in e such that $a_1 A_2 \subseteq A$.

By the same theorem, there are an element $a_2 \in A_2$ and a set $A_3 \subseteq A_2 \setminus \{a_2\}$ in e such that $a_2 A_3 \subseteq A_2$.

This way, setting $A_1 = A$, we choose for each n an element $a_n \in A_n$ and a set $A_{n+1} \subseteq A_n \setminus \{a_n\}$ in e such that $a_n A_{n+1} \subseteq A_n$, as illustrated in the following diagram.

Then $\mathrm{FP}(a_1, a_2, \dots) \subseteq A$: Consider a product $a_{i_1} a_{i_2} \cdots a_{i_n}$ with $i_1 < i_2 < \cdots < i_n$. Each of the following assertions implies the subsequent one.

$$\begin{array}{rcl} a_{i_n} & \in & A_{i_n} \\ a_{i_{n-1}} a_{i_n} & \in & a_{i_{n-1}} A_{i_n} \subseteq A_{i_{n-1}} \\ a_{i_{n-2}} a_{i_{n-1}} a_{i_n} & \in & a_{i_{n-2}} A_{i_{n-1}} \subseteq A_{i_{n-2}} \\ & \vdots & \\ a_{i_1} a_{i_2} \cdots a_{i_n} & \in & a_{i_1} A_{i_2} \subseteq A_{i_1} \subseteq A. \end{array}$$

Awesome, isn't it? □

Had we not insisted that the elements $a_1, a_2, \dots$ are distinct, the conclusion in the Galvin–Glazer Theorem would trivialize in every semigroup with an idempotent $e \in S$. In this case, taking $a_n = e$ for all n, we have that $\mathrm{FP}(a_1, a_2, \dots) = \{e\}$, which is monochromatic but for a trivial reason.

Exercise 1.4. Formulate and prove, using the Compactness Theorem, a finite version of Hindman's Theorem.

As the semigroup $(\mathbb{N}, \cdot)$ is cancellative, for each finite coloring of $\mathbb{N}$ there are also distinct natural numbers $a_1, a_2, \dots$ such that the *finite products* set $\mathrm{FP}(a_1, a_2, \dots)$ is monochromatic.

Definition 1.5. Let S be a semigroup. A set $A \subseteq S$ is an *FP set* if there are distinct elements $a_1, a_2, \dots \in S$ with $\mathrm{FP}(a_1, a_2, \dots) \subseteq A$. In cases where the operation of S is denoted by $+$, we write FS instead of FP.

The proof of the Galvin–Glazer Theorem (Theorem 1.3) establishes the following.

Corollary 1.6. *Every element of a nonprincipal idempotent of βS is an FP set.* □

Theorem 1.7. *For each finite coloring of $\mathbb{N}$, there is a monochromatic set that is both an FS set and an FP set.*

Proof. Let

$$L = \{ p \in \beta\mathbb{N} : \text{each } A \in p \text{ is an FS set} \}.$$

By Corollary 1.6, every idempotent of the semigroup $(\beta\mathbb{N}, +)$ is in L. In particular, the set L is nonempty.

L is a (topologically) closed subset of $\beta\mathbb{N}$: For $p \notin L$, fix $A \in p$ that is not an FS set. Then $p \in [A]$ and $L \cap [A] = \emptyset$.

L is closed under multiplication: Let $p, q \in L$ (in fact, it suffices that $q \in L$) and $A \in pq$. By the Product Characterization Theorem, there are $n \in B \in p$ and $C \in q$ such that $nC \subseteq A$. Pick distinct $c_1, c_2, \dots \in \mathbb{N}$ such that $\mathrm{FS}(c_1, c_2, \dots) \subseteq C$. By distributivity, we always have that $n(c_{i_1} + \cdots + c_{i_m}) = nc_{i_1} + \cdots + nc_{i_m}$, and thus

$$A \supseteq n\,\mathrm{FS}(c_1, c_2, \dots) = \mathrm{FS}(nc_1, nc_2, \dots).$$

This shows that $pq \in L$.

Thus, $(L, \cdot)$ is a subcompany of $(\beta\mathbb{N}, \cdot)$. Let $e = e^2$ be an idempotent in L. Let $A \in e$ be monochromatic for the given coloring. As L and $\mathbb{N}$ are disjoint, we have that $e \in \beta\mathbb{N} \setminus \mathbb{N}$. Thus, the set A is an FP set. Since $e \in L$, we have that A is an FS set. □

THEOREM 1.8. *Let V be an infinite-dimensional vector space over the two-element field $\mathbb{Z}_2$. For each finite coloring of $V \setminus \{\vec{0}\}$, there is an infinite-dimensional subspace U of V such that the set $U \setminus \{\vec{0}\}$ is monochromatic. In particular, V has a monochromatic, infinite-dimensional affine subspace.*

PROOF. $(V, +)$ is a group. By Theorem 5.3, there is an idempotent $e \in \beta V \setminus V$. By the Galvin–Glazer Theorem, there are distinct elements $v_1, v_2, \dots \in V$ such that the set $\mathrm{FS}(v_1, v_2, \dots)$ is monochromatic.

Since the space is over $\mathbb{Z}_2$, we have that $\mathrm{span}\{v_1, v_2, \dots\} = \mathrm{FS}(v_1, v_2, \dots) \cup \{\vec{0}\}$. As spaces over $\mathbb{Z}_2$ spanned by finitely many vectors are finite, the space $\mathrm{span}\{v_1, v_2, \dots\}$ is infinite-dimensional.

For an infinite-dimensional subspace U of V, let $v_1, v_2, \dots \in U$ be linearly independent. Then $v_1 + \mathrm{span}\{v_2, v_3, \dots\} \subseteq U \setminus \{\vec{0}\}$ is an infinite-dimensional affine subspace of V. $\square$

In Theorem 1.8 we cannot request that the whole subspace U is monochromatic.

EXERCISE 1.9. Show that for each vector space there is a 2-coloring such that the only monochromatic subspace is the zero space.

Theorem 1.8 fails for fields other than $\mathbb{Z}_2$.

EXERCISE 1.10. Let V be the countably-infinite-dimensional vector space over a field $\mathbb{F} \neq \mathbb{Z}_2$. Find a 2-coloring of V with no monochromatic infinite-dimensional affine subspace.
Hint: Let $v_1, v_2, \dots$ be linearly independent. For each $v \in V$, represent $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$ with $\alpha_k \neq 0$. Color v green if $\alpha_k = 1$, and red otherwise.

2. Coloring FP sets

We have seen (Corollary 1.6) that every element of a nonprincipal idempotent of βS is an FP set. The converse also holds.

THEOREM 2.1. *Let S be a moving semigroup. Every* FP *set in S belongs to some nonprincipal idempotent of βS. Moreover, for all distinct $a_1, a_2, \dots \in S$, there is an idempotent $e \in \beta S \setminus S$ such that* $\mathrm{FP}(a_n, a_{n+1}, \dots) \in e$ *for all n.*

PROOF. For each n, let $K_n = [\mathrm{FP}(a_n, a_{n+1}, \dots)] \subseteq \beta S$. The set K_n is compact, and therefore so is $K_n \setminus S$. As FP sets are infinite, the set $K_n \setminus S$ is nonempty. Since

$$\beta S \setminus S \supseteq K_1 \setminus S \supseteq K_2 \setminus S \supseteq \cdots,$$

we have by the finite intersection property that the compact set

$$K := \bigcap_{n=1}^{\infty} K_n \setminus S = \bigcap_{n=1}^{\infty} [\mathrm{FP}(a_n, a_{n+1}, \dots)] \setminus S$$

is nonempty. The set K is closed under multiplication: Let $p, q \in K$. In particular, $p, q \notin S$. As S is moving, $\beta S \setminus S$ is a subsemigroup of βS, and thus $pq \notin S$.

Fix n. It remains to verify that $pq \in [\mathrm{FP}(a_n, a_{n+1}, \dots)]$, that is, $\mathrm{FP}(a_n, a_{n+1}, \dots) \in pq$. We will do so using the Product Characterization Theorem. We know that $\mathrm{FP}(a_n, a_{n+1}, \dots) \in p$. For all $a_{i_1} \cdots a_{i_k} \in \mathrm{FP}(a_n, a_{n+1}, \dots)$ (where $n \leq i_1 < \cdots < i_k$), the set $\mathrm{FP}(a_{i_k+1}, a_{i_k+2}, \dots)$ is in q, and

$$a_{i_1} \cdots a_{i_k} \cdot \mathrm{FP}(a_{i_k+1}, a_{i_k+2}, \dots) \subseteq \mathrm{FP}(a_n, a_{n+1}, \dots).$$

Thus, K is a company and there is an idempotent $e \in K$. Then $e \notin S$ and $\mathrm{FP}(a_n, a_{n+1}, \dots) \in e$ for all n. $\square$

COROLLARY 2.2. *Let S be a moving semigroup, and $A_1, \dots, A_k \subseteq S$. If $A_1 \cup \dots \cup A_k$ is an FP set, then there is i such that A_i is an FP set. In particular, for each finite coloring of an FP set, there is a monochromatic FP subset.*

PROOF. Take an idempotent $e \in \beta S \setminus S$ with $A := A_1 \cup \dots \cup A_k \in e$. As e is an ultrafilter, some A_i is in e. As $e \in \beta S \setminus S$ and e is an idempotent, this set A_i is an FP set. □

Given a finite coloring of an FP set $\mathrm{FP}(a_1, a_2, \dots)$, is there necessarily a subsequence $a_{i_1}, a_{i_2}, \dots$ with $\mathrm{FP}(a_{i_1}, a_{i_2}, \dots)$ monochromatic? The answer is negative: Consider the additive semigroup $\mathbb{N}$. Color the even numbers green, and the odd numbers red. Consider $\mathrm{FS}(1, 3, 5, 7, 9, \dots)$. For each subsequence of the odd numbers, every number is red, but every sum of two is green. However, the subset $\mathrm{FS}(1+3, 5+7, 9+11, \dots)$ of $\mathrm{FS}(1, 3, 5, 7, \dots)$ is monochromatic. This illustrates the following theorem.

For finite sets $I, J \subseteq \mathbb{N}$, we write $I < J$ if $i < j$ for all $i \in I$ and $j \in J$, that is, if $\max I < \min J$.

THEOREM 2.3. *Let S be a moving semigroup. Let $a_1, a_2, \dots \in S$ be distinct. For each finite coloring of $\mathrm{FP}(a_1, a_2, \dots)$, there are finite index sets $F_1, F_2, \dots \subseteq \mathbb{N}$ such that:*

1. *$F_1 < F_2 < F_3 < \cdots$.*
2. *The elements $s_n = \prod_{i \in F_n} a_i$, where the indices in the multiplication are taken in increasing order, are distinct.*
3. *The set $\mathrm{FP}(s_1, s_2, \dots)$ (which is contained in $\mathrm{FP}(a_1, a_2, \dots)$) is monochromatic.*

PROOF. Take an idempotent $e \in \beta S \setminus S$ such that $\mathrm{FP}(a_n, a_{n+1}, \dots) \in e$ for all n. As $\mathrm{FP}(a_1, a_2, \dots) \in e$, there is in e a monochromatic subset A_1 of $\mathrm{FP}(a_1, a_2, \dots)$. We repeat the proof of the Galvin–Glazer Theorem, with small changes.

Take $s_1 = \prod_{i \in F_1} a_i \in A_1$ and a set $A_2 \subseteq A_1 \setminus \{s_1\}$ in e such that $s_1 A_2 \subseteq A_1$. Let $n_1 = \max F_1$. As $A_2 \cap \mathrm{FP}(a_{n_1+1}, a_{n_1+2}, \dots) \in e$, we may assume that $A_2 \subseteq \mathrm{FP}(a_{n_1+1}, a_{n_1+2}, \dots)$.

Take $s_2 = \prod_{i \in F_2} a_i \in A_2$ and a set $A_2 \subseteq A_2 \setminus \{s_2\}$ in e such that $s_2 A_3 \subseteq A_2$. Let $n_2 = \max F_2$. We may assume that $A_3 \subseteq \mathrm{FP}(a_{n_2+1}, a_{n_2+2}, \dots)$.

Continue in the same manner.

Then the set $\mathrm{FP}(s_1, s_2, \dots)$ is a subset of A, and is therefore monochromatic. □

EXERCISE 2.4. Prove Theorem 2.3 under the more general assumption, that, for some subsequence of the given sequence $a_1, a_2, \dots$, the subsemigroup generated by this subsequence is moving.

Let $[\mathbb{N}]^{<\infty}$ denote the set of all finite subsets of $\mathbb{N}$. If, in the following theorem, we do not insist that the sets F_n are disjoint, then its conclusion becomes trivial: In every increasing chain of finite sets there is a monochromatic sub-chain, and chains are closed under finite unions.

COROLLARY 2.5. *For each finite coloring of the set $[\mathbb{N}]^{<\infty}$ there are finite sets $F_1, F_2, \dots \subseteq \mathbb{N}$ such that $F_1 < F_2 < \dots$, and all sets*

$$F_{i_1} \cup F_{i_2} \cup \dots \cup F_{i_m}$$

(for $i_1 < i_2 < \dots < i_m$) are of the same color.

PROOF. The semigroup $([\mathbb{N}]^{<\infty}, \cup)$ is moving, and $[\mathbb{N}]^{<\infty} \setminus \{\emptyset\} = \mathrm{FP}(\{1\}, \{2\}, \dots)$, an FP set. Apply Theorem 2.3. □

The semigroup $([\mathbb{N}]^{<\infty}, \cap)$ is not moving, and does not have any moving subsemigroup. An example similar to that in Exercise 1.9 shows that, in the last corollary, one cannot replace $\cup$ by $\cap$.

3. Application to systems of polynomial inequalities

A semigroup $(S,+)$ is *abelian* if $a+b=b+a$ for all $a,b\in S$. Usually, the operator of an ablian semigroup is denoted "+".

DEFINITION 3.1. A *ring* is a structure $(R,+,\cdot)$ such that $(R,+)$ is an abelian group, $(R,\cdot)$ is a semigroup with an identity element (a monoid), and the distributive law holds:

$$\begin{aligned} a(b+c) &= ab+ac \\ (b+c)a &= ba+ca \end{aligned}$$

for all $a,b,c\in R$. A *monomial* (in x) over R is an expression of the form

$$a_1x^{d_1}a_2x^{d_2}\cdots a_mx^{d_m}a_{m+1},$$

where $a_1,\dots,a_{m+1}\in R$ and $d_1,\dots,d_m\in\mathbb{N}$. The *degree* of a monomial is the sum of powers of x occurring in the monomial. A *polynomial* (in x) over R is a sum of monomials (in x, over R). The *degree* of a polynomial is the maximum degree of a monomial in the polynomial. $R[x]$ is the family of all polynomials in x over R.

When the ring R is a field, multiplication is abelian, and every monomial is of the form ax^d. Thus, in this case, the notion of polynomial coincides with the familiar one. An example of a ring with nonabelian multiplication is the ring of all square $n\times n$ matrices over a fixed field.

The product of monomials is a monomial and the sum of polynomials is a polynomial. By the distributive law, for a ring R, the product of polynomials is a polynomial, and thus $R[x]$ is also a ring.

DEFINITION 3.2. A *system of polynomial inequalities* over a ring R is a system of inequalities

$$\begin{aligned} f_1(x) &\neq g_1(x) \\ f_2(x) &\neq g_2(x) \\ &\vdots \\ f_m(x) &\neq g_m(x), \end{aligned}$$

where $f_1(x),\dots,f_m(x),g_1(x),\dots,g_m(x)\in R[x]$. A *solution* to such a system is an element $a\in R$ satisfying all inequalities in the system, that is, such that $f_i(a)\neq g_i(a)$ for all $i=1,\dots,m$.

Every system of polynomial inequalities can be brought to the form

$$\begin{aligned} f_1(x) &\neq 0 \\ f_2(x) &\neq 0 \\ &\vdots \\ f_m(x) &\neq 0, \end{aligned}$$

with $f_1(x),\dots,f_m(x)\in R[x]$. Indeed, replace each inequality $f_i(x)\neq g_i(x)$ by the inequality $f_i(x)-g_i(x)\neq 0$. Thus, we will restrict attention to inequalities of this simplified form.

In the infinite ring

$$\mathbb{Z}_2^{\mathbb{N}}=\{\,(a_1,a_2,\dots):a_1,a_2,\dots\in\mathbb{Z}_2\,\},$$

with coordinate-wise addition and multiplication, there are polynomial inequalities with no solution, e.g., $x^2-x\neq 0$. We will see, though, that if a system of polynomial inequalities has a solution, then it has many solutions.

LEMMA 3.3. *Let $f(x)\in R[x]$ be a polynomial of degree n. For each constant $a\in R$, there is $g(x)\in R[x]$, of degree smaller than n, such that $f(a+x)=f(x)+g(x)$.*

Proof. It suffices to prove the claim for monomials, and this follows by opening brackets. ☐

Exercise 3.4. Prove Lemma 3.3.

Definition 3.5. For elements $a_1, \dots, a_n$ is a ring R, let $\mathrm{FS}(a_1, a_2, \dots, a_n)$ be the set of all (finite) sums $a_{i_1} + \cdots + a_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$, $k \leq n$ arbitrary.

Lemma 3.6. *Let $f(x) \in R[x]$ be a polynomial of degree n. If there are $a_1, \dots, a_{n+1} \in R$ such that $f(a) = 0$ for all $a \in \mathrm{FS}(a_1, \dots, a_{n+1})$, then $f(0) = 0$.*

Proof. By induction on n, the degree of $f(x)$.

$n = 1$: By Lemma 3.3, there is $c = g(x) \in R[x]$ of degree smaller than 1, that is, a constant, such that

$$f(a_1 + x) = f(x) + g(x) = f(x) + c.$$

Substitute a_2 for x, to obtain $0 = f(a_1+a_2) = f(a_2)+c = c$. Thus, $f(a_1+x) = f(x)$. Substitute 0 for x, to obtain $0 = f(a_1) = f(0)$.

$n > 1$: By Lemma 3.3, there is $g(x) \in R[x]$, of degree smaller than n, such that $f(a_1 + x) = f(x) + g(x)$. For each $a \in \mathrm{FS}(a_2, \dots, a_n)$, we have that

$$0 = f(a_1 + a) = f(a) + g(a) = 0 + g(a) = g(a).$$

By the inductive hypothesis, $g(0) = 0$, and thus $0 = f(a_1) = f(a_1 + 0) = f(0) + g(0) = f(0)$. The proof is completed. ☐

Theorem 3.7. *Let R be an infinite ring. If a system of polynomial inequalities over R has a solution, then it has infinitely many solutions. Moreover, for each solution a_0, the set of solutions is of the form $a_0 + A$, where A is an FS set.*

Proof. Fix a polynomial system of inequalities. Let $f_1(x), \dots, f_m(x) \in R[x]$ be the polynomials in this system, and let A be its nonempty solution set. Fix $a_0 \in A$, and consider the new system

$$\begin{aligned} f_1(a_0 + x) &\neq 0 \\ f_2(a_0 + x) &\neq 0 \\ &\vdots \\ f_m(a_0 + x) &\neq 0. \end{aligned}$$

Let A' be its solution set. Then $0 \in A'$, and $a_0 + A' = A$. We may assume, thus, that this is the situation in the original system, that is, that $0 \in A$, and prove that A is an FS set.

For $f(x) \in R[x]$, let $f^{-1}(0) = \{\, a \in R : f(a) = 0 \,\}$. As $A = f_1^{-1}(0)^{\mathsf{c}} \cap \cdots \cap f_m^{-1}(0)^{\mathsf{c}}$ (all complements taken in R), we have that

$$f_1^{-1}(0) \cup \cdots \cup f_m^{-1}(0) = A^{\mathsf{c}}.$$

Since $(R, +)$ is a group and R is an FS set, we have by Corollary 2.2 that either A or A^{c} is an FS set. Assume, towards a contradiction, that A^{c} is an FS set. Then, by the same corollary, there is i such that $f_i^{-1}(0)$ is an FS set. By Lemma 3.6, we have that $0 \in f_i^{-1}(0)$, and thus $0 \notin A$; a contradiction. ☐

Exercise 3.8. Prove that, in general, we may not request that in the last theorem that the solution set is an FS set.
Hint: Consider the inequality $x^2 \neq x$ over the ring $\mathbb{Z}_3 \times \mathbb{Z}_2^{\mathbb{N}}$, with coordinate-wise addition and multiplication.

4. Comments for Chapter 4

Hindman's Theorem is a landmark in combinatorial number theory and Ramsey theory. The finite version of Hindman's Theorem (Exercise 1.4) is due, independently, to Jon Folkman (1965, unpublished), Jon H. Sanders (*A Generalization of Schur's Theorem*, Doctoral Dissertation, Yale University, 1968), Richard Rado (*Some partition theorems*, Colloquia Mathematica Societatis Janos Bolyai, 1969), and Vladimir I. Arnautov (*Nondiscrete topologizability of countable rings*, Soviet Mathematics Doklady, 1970). Of course, this results was originally proved without the help of the later Hindman's Theorem. Sanders in his dissertation, and later Ron L. Graham and Bruce L. Rothschild (*Ramsey's theorem for n-parameter sets*, Transactions of the American Mathematical Society, 1971), posed Hindman's Theorem as a conjecture.

The celebrated mathematician David Hilbert has in fact proved an infinitary weak form of Hindman's Theorem, namely: For each finite coloring of $\mathbb{N}$ and each m, there are natural numbers $a_1, a_2, \dots, a_m$ and an infinite set $B \subseteq \mathbb{N}$ such that the set

$$\{\, a+b : a \in \mathrm{FS}(a_1, a_2, \dots, a_m), b \in B \,\}$$

is monochromatic. Naturally, here $\mathrm{FS}(a_1, a_2, \dots, a_m)$ is the set of all (necessarily, finite) sums of distinct members of the set $a_1, a_2, \dots, a_m$ (*Über die Irreduzibilitat ganzer rationaler Funktionen mit ganzzahlingen Koefzienten*, Journal für die reine und angewandte Mathematik, 1892).

Hindman's original proof (*Finite sums from sequences within cells of a partition of* $\mathbb{N}$, Journal of Combinatorial Theory (A), 1974), while elementary, was very involved. Hindman himself commented that he "does not understand it". The natural proof provided here is due to Fred Galvin and Steve Glazer. Galvin realized that all one needs is an ultrafilter as in Theorem 6.8, and Glazer realized that idempotents in $\beta\mathbb{N}$ have this property.

FS sets were introduced by Hillel Furstenberg in his book *Recurrence in ergodic theory and combinatorial number theory* (Princeton University Press, 1981). There, FS sets are requested to be *equal* to sets of the form $\mathrm{FS}(a_1, a_2, \dots)$, but we follow the terminological convention of the Hindman–Strauss monograph. In Furstenberg's book, and in many later sources, FS sets are called *IP sets*. Thus, we have seen that elements of idempotents in $\beta\mathbb{N}$ are IP sets. Amusingly, the initials "IP" do not stand for "idempotents", but rather for "infinite-dimensional parallelepiped", since the finite sums of a set of three linearly independent vectors in space form the vertices of an (three-dimensional) parallelepiped.

Theorem 1.7 asserts that there are distinct numbers $k_1, k_2, \dots$ and distinct numbers $n_1, n_2, \dots$ such that the set $\mathrm{FS}(k_1, k_2, \dots) \cup \mathrm{FP}(n_1, n_2, \dots)$ is monochromatic. But this theorem does *not* assert that these sequences may be chosen to be identical. Indeed, Hindman has found a concrete finite coloring (in fact, a 7-coloring) of $\mathbb{N}$ such that there are *no* distinct $k_1, k_2, \dots$ such that the set $\mathrm{FS}(k_1, k_2, \dots) \cup \mathrm{FP}(k_1, k_2, \dots)$ is monochromatic, even if we consider only sums and products of pairs of elements! Using this fact, he proved that there is no ultrafilter p on $\mathbb{N}$ satisfying $p + p = p \cdot p$. Details are available in the Hindman–Strauss monograph.

As already observed, the Galvin–Glazer Theorem (Theorem 1.3) does not hold for all semigroups. However, if S has a subsemigroup T for which this coloring theorem holds, then the theorem holds for S too, since any coloring of S is also a coloring of T. Thus, Shevrin's classification of the necessary subsemigroups of infinite semigroups (See comments to Chapter 3) is relevant. We obtain the following theorem, pointed out in a joint work with Gili Golan, *Hindman's coloring theorem in arbitrary semigroups*, Journal of Algebra, 2013.

THEOREM 4.1. *Let S be an arbitrary semigroup. For each finite coloring of S there are distinct $a_1, a_2, \dots \in S$ and a finite set $F \subseteq S$ such that the (infinite) set $\mathrm{FP}(a_1, a_2, \dots) \setminus F$ is monochromatic.* $\square$

EXERCISE 4.2. Prove Theorem 4.1.

Hint: By the above discussion, it suffices to consider cases (6) and (7) in Shevrin's classification.

An infinite set is *almost-monochromatic* for a given coloring if all but finitely many members of that set have the same color. In the above-cited joint work with Golan, the semigroups possessing, for each finite coloring, an infinite almost-monochromatic subsemigroup are characterized.

Lemma 3.3, Lemma 3.6 and Theorem 3.7, modulo the assertion that the solution set is a translate of an FS set, are proved in Arnautov's cited paper. The proof of Theorem 3.7 provided here is due to Hromulyak, Protasov and Zelenyuk, *Topologies on countable groups and rings*, Doklady Akademia Nauka Ukraine, 1991.

A *ring topology* on a ring R is a topology on R such that addition and multiplication are continuous. By continuity of the ring operations, every solution set of a systems of polynomial inequalities is open in any ring topology. Such solution sets are known as *Zariski open* or *unconditionally open*. If 0 is a unique solution of a system of polynomial inequalities, then every ring topology is discrete. The Zariski open sets may not be a basis for a topology, since they often do not separate points, that is the Hausdorff property fails. Andrey A. Markov (*On unconditionally closed sets*, Matematicheskii Sbornik, 1946) proved that if 0 is *not* a unique solution of a system of polynomial inequalities in a countably infinite ring R, then there is a nondiscrete, Hausdorff ring topology on R. It follows from Arnautov's Theorem (Theorem 3.7) that on every countably-infinite ring there is a nondiscrete ring topology.

For the present paragraph only, we do not request a topological space to be Hausdorff. A *ring topology* on a ring R is a topology on R such that addition and multiplication are continuous. By continuity of the ring operations, every solution set of a systems of polynomial inequalities is open in any ring topology, and these sets form a basis for a topology, known as the *Zariski topology*. If 0 is a unique solution of a system of polynomial inequalities, then the Zariski topology is discrete. The Zariski topology is often non-Hausdorff, but Andrey A. Markov (*On unconditionally closed sets*, Matematicheskii Sbornik, 1946) proved that if 0 is *not* a unique solution of a system of polynomial inequalities in a countably infinite ring R, then there is a nondiscrete, Hausdorff ring topology on R. It follows from Arnautov's Theorem (Theorem 3.7) that on every infinite countable ring there is a nondiscrete Hausdorff ring topology. This consequence was first proved, by direct means, by Arnautov in his above-cited paper.

CHAPTER 5

Monochromatic arithmetic progressions

1. Ideals in semigroups

Definition 1.1. Let S be a semigroup. A *left ideal* (of S) is a nonempty set $L \subseteq S$ such that $SL := \{ sa : s \in S, a \in L \} \subseteq L$, that is, $sL := \{ sa : a \in L \} \subseteq L$ for all $s \in S$. A left ideal L is *minimal* if no left ideal of S is properly contained in S. Elements of minimal left ideals are called *minimal elements*.

It follows that every left ideal of a semigroup S is a subsemigroup of S, and that, for each $a \in S$, the set Sa is a left ideal of S.

Lemma 1.2. *Let S be a semigroup.*

(1) *If a is a minimal element, then the element sa is minimal for each $s \in S$.*
(2) *For each minimal left ideal L and every $a \in L$, we have that $Sa = L$.*
(3) *For each minimal element $a \in S$, the set Sa is a minimal left ideal.*
(4) *If S is a company, then every minimal left ideal of S is compact.*

Proof. (1) Let L be a minimal left ideal containing a. Then $sa \in L$.
(2) $Sa \subseteq L$ is a left ideal. By minimality of L, $Sa = L$.
(3) By (2).
(4) By (2), the minimal left ideal is of the form Sa. As right multiplication by a is a continuous function and S is compact, the left ideal Sa is compact. □

Lemma 1.3 (Fixing). *Let S be a semigroup. Let $a \in S$. The following assertions are equivalent:*

(1) *a is minimal.*
(2) *For each $b \in S$, there is $c \in S$ such that $cba = a$.*

Proof. (1) ⇒ (2): Let L be a minimal left ideal with $a \in L$. Then $ba \in L$, and therefore $L = Sba$. As $a \in L$, there is $c \in S$ such that $a = cba$.

(2) ⇒ (1): Let L be a left ideal with $L \subseteq Sa$. Take $ba \in L$. Then there is c such that $a = cba \in L$. Thus, $Sa \subseteq L$, and therefore $L = Sa$. This shows that Sa is a minimal left ideal and $a \in L = Sa$. □

An element a of a semigroup S is a *minimal idempotent* element if it is both a minimal element and an idempotent.

Lemma 1.4. *Let S be a company. Every left ideal of S contains a minimal idempotent element.*

Proof. We first show that every left ideal of S contains a minimal left ideal. Let L be a left ideal. Fix an element $a \in L$. Then $Sa \subseteq L$, and Sa is a compact left ideal. Thus, the family of all compact left ideals contained in L is nonempty. By the finite intersection property of compact sets, this family satisfies the conditions of Zorn's Lemma, and therefore has a minimal element M. Let $I \subseteq M$ be a left ideal. Then, for any $b \in I$, Sb is a compact left ideal contained in I. By minimality of M, we have that $Sb = I = M$.

Being a minimal left ideal in a company, the set M is compact and thus a company. Thus, there is an idempotent in M. $\square$

Minimal elements are minimal in every company where they belong.

Lemma 1.5. *Let S be a company and $a \in S$ a minimal element. For each subcompany T of S such that $a \in T$, the element a is also minimal in T.*

Proof. The set Ta is a left ideal of T. Take a minimal idempotent $e \in Ta$. Then there is an element $t \in T$ with $e = ta$. By the Fixing Lemma, there is $s \in S$ such that $se = a$. Then $a = se = see = ae$. As e is minimal in T and $a \in T$, the element $a = ae$ is also minimal in T. $\square$

Definition 1.6. An *ideal* (of S) is a nonempty set $I \subseteq S$ with $IS, SI \subseteq I$.

Lemma 1.7. *Let I be an ideal of a semigroup S. For each minimal element a, we have that $a \in I$.*

Proof. Let $b \in I$. By the Fixing Lemma, there is $c \in S$ such that $a = cba \in I$. $\square$

Let S be a semigroup and m be a natural number. For visual clarity, we present elements of S^m as columns. The set S^m is a semigroup with the coordinate-wise product:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} := \begin{pmatrix} a_1 b_1 \\ \vdots \\ a_m b_m \end{pmatrix}.$$

Elements of S^m will be denoted by boldface letters: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$.

Lemma 1.8. *Let S be a semigroup and $a_1, \dots, a_m \in S$ be minimal. Then the element*

$$\mathbf{a} := \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

is minimal in S^m.

Proof. By the Fixing Lemma, it suffices to prove that for each $\mathbf{b} \in S^m$ there is $\mathbf{c} \in S^m$ such that $\mathbf{cba} = \mathbf{a}$. Let

$$\mathbf{ba} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} b_1 a_1 \\ \vdots \\ b_m a_m \end{pmatrix}.$$

By the Fixing Lemma, there is for each $i = 1, \dots, m$ an element $c_i \in S$ such that $c_i b_i a_i = a_i$. Let

$$\mathbf{c} := \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}.$$

Then

$$\mathbf{cba} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \cdot \begin{pmatrix} b_1 a_1 \\ \vdots \\ b_m a_m \end{pmatrix} = \begin{pmatrix} c_1 b_1 a_1 \\ \vdots \\ c_m b_m a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \mathbf{a}. \qquad \square$$

Recall that, for a company S, the product topology on S^m is the one with the basic open sets

$$U_1 \times \cdots \times U_m,$$

where $U_1, \dots, U_m \subseteq S$.

EXERCISE 1.9. Let S be a company. The set S^m, with the product topology and coordinatewise multiplication, is a company.

2. van der Waerden's Theorem

The following theorem, due to Bartel L. van der Waerden, proves a conjecture of Schur.

THEOREM 2.1 (van der Waerden). *For each finite coloring of* $\mathbb{N}$*, there are arbitrarily long monochromatic arithmetic progressions.*

Towards the proof of van der Waerden's Theorem, assume that $\mathbb{N}$ is finitely colored. Fix a natural number m. We wish to find a monochromatic arithmetic progression of length m. It suffices to find an ultrafilter $p \in \beta\mathbb{N}$ such that every element of p contains an arithmetic progression of length m. Explicitly, fix $A \in p$. We need that there are $a, d \in \mathbb{N}$ such that $a, a+d, \dots, a+(m-1)d \in A$, that is,

$$\begin{pmatrix} a \\ a+d \\ \vdots \\ a+(m-1)d \end{pmatrix} \in [A]^m.$$

The set $[A]^m$ is a neighborhood of the element

$$\begin{pmatrix} p \\ \vdots \\ p \end{pmatrix}$$

in the product topology of $(\beta\mathbb{N})^m$.

Consider the following two subsets of $\mathbb{N}^m$. These sets are, in particular, subsets of $(\beta\mathbb{N})^m$:

$$\begin{aligned} \mathrm{AP} &:= \left\{ \begin{pmatrix} a \\ a+d \\ \vdots \\ a+(m-1)d \end{pmatrix} : a, d \in \mathbb{N} \right\}; \\ \mathrm{AP}_0 &:= \mathrm{AP} \cup \left\{ \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} : a \in \mathbb{N} \right\}. \end{aligned}$$

EXERCISE 2.2. For each $\mathbf{s} \in \mathbb{N}^m$, The function on $(\beta\mathbb{N})^m$, defined by $\mathbf{q} \mapsto \mathbf{s}+\mathbf{q}$, is continuous.

LEMMA 2.3. $\overline{\mathrm{AP}_0}$ *is a subcompany of* $(\beta\mathbb{N})^m$.

PROOF. As every sum of two arithmetic progressions is an arithmetic progression, the set AP_0 is a subsemigroup of $\mathbb{N}^m$. The lemma follows by continuity: Let $\mathbf{p}, \mathbf{q} \in \overline{\mathrm{AP}_0}$. We need to show that $\mathbf{p}+\mathbf{q} \in \overline{\mathrm{AP}_0}$, that is, every neighborhood of $\mathbf{p}+\mathbf{q}$ intersects AP_0.

Let U be a neighborhood of $\mathbf{p}+\mathbf{q}$. By right continuity, there is a neighborhood V of $\mathbf{p}$ such that $V+\mathbf{q} \subseteq U$. As $\mathbf{p} \in \overline{\mathrm{AP}_0}$, there is $\mathbf{s} \in V \cap \mathrm{AP}_0$. In particular, $\mathbf{s}+\mathbf{q} \in U$. By continuity of left multiplication by an element of $\mathbb{N}^m$, there is a neighborhood W of $\mathbf{q}$ such that $\mathbf{s}+W \subseteq U$. As $\mathbf{q} \in \overline{\mathrm{AP}_0}$, there is $\mathbf{t} \in W \cap \mathrm{AP}_0$ and, in particular $\mathbf{s}+\mathbf{t} \in U$. As $\mathbf{s}$ and $\mathbf{t}$ are in AP_0, so is their sum. Thus, $\mathbf{s}+\mathbf{t} \in U \cap \mathrm{AP}_0$ and the latter intersection is nonempty. $\square$

LEMMA 2.4. *Let $p \in \beta\mathbb{N}$, and*

$$\mathbf{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix}.$$

Then:

(1) *Every neighborhood of $\mathbf{p}$ contains one of the form $[A]^m$, with $A \in p$.*
(2) $\mathbf{p} \in \overline{\mathrm{AP}_0}$.

PROOF. (1) Let $[A_1] \times \cdots \times [A_m]$ be a basic open neighborhood of $\mathbf{p}$ contained in the given neighborhood. As $A_1, \cdots, A_m \in p$, we have that $A := A_1 \cap \cdots \cap A_m \in p$. Then $\mathbf{p} \in [A]^m \subseteq [A_1] \times \cdots \times [A_m]$.

(2) It suffices to consider neighborhoods of the form (1). As $A \in p$, there is $a \in A$. Then

$$\begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \in [A]^m \cap \mathrm{AP}_0.$$ □

LEMMA 2.5. *The set $\overline{\mathrm{AP}}$ is an ideal of $\overline{\mathrm{AP}_0}$.*

PROOF. The set AP is an ideal of AP_0. The assertion follows, by continuity considerations as in the proof of Lemma 2.3. □

EXERCISE 2.6. Prove Lemma 2.5.

LEMMA 2.7. *Let $p \in \beta\mathbb{N}$ be a minimal element. Then*

$$\mathbf{p} := \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in \overline{\mathrm{AP}}.$$

PROOF. We collect the information gained thus far: By Lemma 1.8, The element $\mathbf{p}$ is minimal in $(\beta\mathbb{N})^m$. Since $\mathbf{p} \in \overline{\mathrm{AP}_0}$, it is also minimal in this subcompany. As $\overline{\mathrm{AP}}$ is an ideal of $\overline{\mathrm{AP}_0}$, the minimal element $\mathbf{p}$ is in $\overline{\mathrm{AP}}$ (Lemma 1.7). □

We can now conclude the proof of van der Waerden's Theorem. Given a finite coloring of $\mathbb{N}$, take a minimal element $p \in \beta\mathbb{N}$ and a monochromatic set $A \in p$. By the last lemma, we have that $\mathbf{p} \in \overline{\mathrm{AP}}$, and thus its neighborhood $[A]^m$ intersects the set AP. Therefore, there is

$$\begin{pmatrix} a \\ a+d \\ \vdots \\ a+(m-1)d \end{pmatrix} \in [A]^m$$

with $a, d \in \mathbb{N}$. Then the elements $a, a+d, \dots, a+(m-1)d$ are in A, and are thus of the same color.

EXERCISE 2.8. Find a 2-coloring of $\mathbb{N}$ with no infinite monochromatic arithmetic progression.

EXERCISE 2.9. Prove the following finite version of van der Waerden's Theorem: Let k and m be natural numbers. There is n such that, for each k-coloring of $\{1, \dots, n\}$, there is a monochromatic arithmetic progression of length m in the colored set.

The proof of van der Waerden's Theorem shows that for each $p \in \beta\mathbb{N}$ minimal in $(\beta\mathbb{N}, +)$, every element $A \in p$ contains arbitrarily long arithmetic progressions. Similarly, if p is minimal in $(\beta\mathbb{N}, \cdot)$, then every element $A \in p$ contains arbitrarily long *geometric* progressions $a, aq, \dots, aq^{m-1}$. The following theorem is stronger.

THEOREM 2.10. *For each finite coloring of $\mathbb{N}$, there is a color with arbitrarily long arithmetic and geometric progressions of that color.*

PROOF. The proof is similar to the proof of Theorem 1.7. Say that a set $A \subseteq \mathbb{N}$ is an *AP set* if there are arbitrarily long arithmetic progressions in A. Let

$$L = \{\, p \in \beta\mathbb{N} : \text{each } A \in p \text{ is an AP set} \,\}.$$

We have seen that every minimal element p in $(\beta\mathbb{N}, +)$ is in L. It is easy to see that L is a left ideal of the company $(\beta\mathbb{N}, \cdot)$. Let $p \in L$ be a minimal element of $(\beta\mathbb{N}, \cdot)$.

Take a monochromatic set $A \in p$. By minimality of p, there are arbitrarily long geometric progressions in A. As $p \in L$, there are arbitrarily long arithmetic progressions in A. $\square$

EXERCISE 2.11. Complete the proof of Theorem 2.10, by showing that L is a left ideal of the company $(\beta\mathbb{N}, \cdot)$.

EXERCISE 2.12. Prove that, in Theorem 2.10, we may request that there are, in addition, FS and FP sets of the same color.
Hint: In the definition of L, request that every A is AP and FS. Prove that L is nonempty. Prove that it is a left ideal of $(\beta\mathbb{N}, \cdot)$. Take a minimal idempotent in L.

3. Excursion: the game EquiDist

We introduce a two-player game based on a concrete realization of van der Waerden's Theorem. We describe here its simplest variation. Additional variations are easy to come up with, search "EquiDist game" online for some examples.

By van der Waerden's Theorem and the Compactness Theorem, we know that there is a natural number N such that for each 2-coloring of the numbers $1, 2, \dots N$ there is a monochromatic arithmetic progression of length 3.

PROPOSITION 3.1. *For each coloring of the numbers 1,2,3,4,5,6,7,8 and 9 in red and green, there is a monochromatic 3-element arithmetic progression.*

PROOF. Assume that we are given a coloring with no monochromatic 3-element arithmetic progression. We may assume that the color of 5 is red.

Assume that 3 or 7 is red. Then, by symmetry, we may assume that 3 is red. Then 1, 4 and 7 green:

$$1 \quad 2 \quad {\color{red}3} \quad 4 \quad {\color{red}5} \quad 6 \quad 7 \quad 8 \quad 9\,.$$

We obtain a green arithmetic progression:

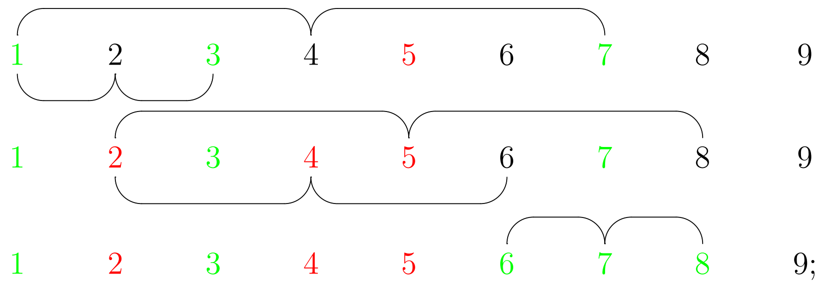
$$ {\color{green}1} \quad 2 \quad {\color{red}3} \quad {\color{green}4} \quad {\color{red}5} \quad 6 \quad {\color{green}7} \quad 8 \quad 9\,;$$

a contradiction.

Thus, we the coloring of 3, 5 and 7 must be of the following form:

$$1 \quad 2 \quad {\color{green}3} \quad 4 \quad {\color{red}5} \quad 6 \quad {\color{green}7} \quad 8 \quad 9$$

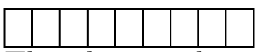
The numbers 1,5, and 9 cannot all be red. Thus, 1 or 9 is green. By symmetry, we may assume that 1 is green. Then:



a contradiction. □

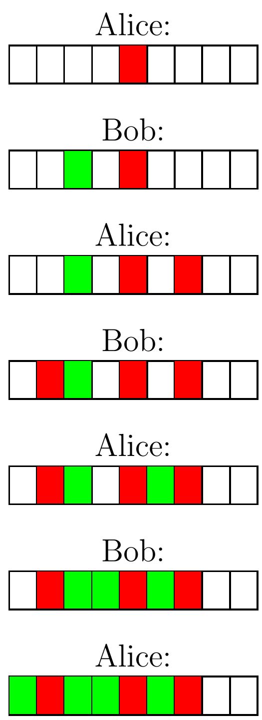
Exercise 3.2. Find a 2-coloring of the numbers 1,2,3,4,5,6,7,8 with no monochromatic 3-element arithmetic progression.

We now describe the game. The game board looks as follows.



The players, that will be called *Alice* and *Bob*, have red and green pieces. Each player, in turn, places a red or a green piece on an empty cell. The first player to place a piece such that there are three equidistant pieces of the same color looses.

Following is an example of a play:



Bob looses, since placing a piece of any color in any cell would result in three equidistant pieces of the same color.

As in Tic-Tac-Toe, one may play the game without the board and pieces, by drawing the board on a page and writing X and O in the cells instead of placing there pieces.

Exercise 3.3. Consider the variation of EquiDist where Alice is only allowed to use red pieces, and Bob is only allowed to place green ones. Prove that Bob has a winning strategy.

4. Comments for Chapter 5

van der Waerden's Theorem (Theorem 2.1) was first conjectured by Issai Schur and later, independently, by Pierre J. H. Baudet.

The proof idea of van der Waerden's Theorem, spanned through the first and second sections, is due to Hillel Furstenberg and Yitzhak Katznelson (*Idempotents in compact semigroups and Ramsey Theory*, Israel Journal of Mathematics, 1989). Furstenberg and Katznelson used the language of topological dynamical systems and enveloping semigroups. Their proof was converted to the one included here by Vitaly Bergelson and Neil Hindman (*Nonmetrizable topological dynamics and Ramsey Theory*, Transactions of the American Mathematical Society, 1990). Furstenberg's original proof was in the language of dynamical systems. According to the Hindman–Strauss monograph, this proof shows

> how much one can get for how little ... It is enough to make someone raised on the work ethic feel guilty.

Theorem 2.10 is due to Vitaly Bergelson and Neil Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Transactions of the American Mathematical Society, 1990. It is easy to see that the set L in the proof of Theorem 2.10 is in fact an ideal. Not only of $(\beta\mathbb{N}, \cdot)$, but also of $(\beta\mathbb{N}, +)$. The existence of this simultaneous ideal was first observed in the Hindman–Strauss monograph (Theorem 14.5).

I have suggested this game to my family in 2011. My son, Avraham Tsaban, and my nephew, Ariel Vishne, were, respectively, 12 and 18 years old then. They soon came up with two results: Avraham came up with Exercise 3.3, and Ariel proved, by exhaustive computer search, that Bob has a winning strategy in (9-cell) EquiDist. Ariel found out that whom has a winning strategy depends on the number of cells. A proof of Vishne's observation not using computers is unknown to us.

CHAPTER 6

Monochromatic words with a wildcard character

1. Colorings of the free semigroup

Throughout this chapter, we use the following conventions.

DEFINITION 1.1. F denotes a nonempty finite set. The elements of the set F will be called *letters*, and the set F will be called the *alphabet set*. A *word* in the alphabet set F is a finite sequence of letters from F.

The *concatenation* of words $u = a_1a_2 \dots a_m$ and $w = b_1b_2 \dots b_k$ in the alphabet set F is the word $uw = a_1a_2 \dots a_m b_1 b_2 \dots b_k$. Σ is the set of all words in the alphabet set F. The set Σ, with the concatenation operation, is the *free semigroup* in the alphabet set F.

For a variable x, the set Σ_x is the free semigroup in the alphabet set $F \cup \{x\}$.

At least one of the letters w_i in a word $w(x) = w_1w_2 \dots w_k \in \Sigma_x \setminus \Sigma$ is the variable x. We treat the variable x a wildcard character, that is, such that every letter from F may be substituted for x to obtain a word in Σ. For example, if $\Sigma = \{1, 2, 3\}$ and $w(x) = 103x545xx43$, then

$$
\begin{aligned}
w(1) &= 103\mathbf{1}545\mathbf{11}43, \\
w(2) &= 103\mathbf{2}545\mathbf{22}43 \text{ and} \\
w(3) &= 103\mathbf{3}545\mathbf{33}43
\end{aligned}
$$

are all words in Σ.

The following proof is very similar to that of van der Waerden's Theorem.

THEOREM 1.2 (Hales–Jewett). *Let $F = \{a_1, \dots, a_m\}$ be a finite nonempty alphabet set, and let Σ be the free semigroup in the alphabet set F. For each finite coloring of Σ there is a word $w(x) \in \Sigma_x \setminus \Sigma$ such that all words $w(a_1), w(a_2), \dots, w(a_m)$ have the same color.*

PROOF. Let p be a minimal element in $\beta\Sigma$. Let $A \in p$ be a monochromatic set. We will prove that there is a word $w(x) \in \Sigma_x \setminus \Sigma$ such that

$$w(a_1), w(a_2), \dots, w(a_m) \in A.$$

Let

$$\mathbf{p} := \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\Sigma)^m.$$

By Lemma 1.8 the element $\mathbf{p}$ is minimal in $(\beta\Sigma)^m$. By Lemma 1.5, this element is minimal in every subcompany of $(\beta\Sigma)^m$ where it belongs. The set

$$T = \left\{ \begin{pmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_m) \end{pmatrix} : w(x) \in \Sigma_x \right\}$$

is a subsemigroup of Σ^m. By continuity considerations, the set $\overline{T}$ is a subsemigroup of $(\beta\Sigma)^m$. For each word $w \in \Sigma$, we have that

$$\mathbf{w} := \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} \in T,$$

and thus $\mathbf{p} \in \overline{T}$. In summary, the vector $\mathbf{p}$ is minimal in $\overline{T}$.

The set

$$I = \left\{ \begin{pmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_m) \end{pmatrix} : w(x) \in \Sigma_x \setminus \Sigma \right\}$$

is an ideal of T, and by continuity considerations, the set $\overline{I}$ is an ideal of $\overline{T}$. Thus, $\mathbf{p} \in \overline{I}$ (Lemma 1.7). As $A \in p$, there is

$$\begin{pmatrix} w(a_1) \\ \vdots \\ w(a_m) \end{pmatrix} \in [A]^m \cap I,$$

where $w(x) \in \Sigma_x \setminus \Sigma$. Then $w(a_1), w(a_2), \dots, w(a_m) \in A$. $\square$

Exercise 1.3. Prove the following assertions, made in the proof of the Hales–Jewett Theorem:

(1) $\overline{T}$ is a subsemigroup of $(\beta\Sigma)^m$.
(2) $\overline{I}$ is an ideal of $\overline{T}$.

The remainder of this chapter is dedicated to applications of the Hales–Jewett Theorem.

2. Monochromatic homothetic copies

van der Waerden's follows easily from the Hales–Jewett Theorem: Let c be a finite coloring of $\mathbb{N}$ and m be a natural number. Take the alphabet set $F = \{1, \dots, m\}$, and let Σ be the free semigroup over F. Define a finite coloring χ of Σ by

$$\chi(s_1 s_2 \dots s_k) := c(s_1 + s_2 + \cdots + s_k)$$

for all $s_1, s_2, \dots, s_k \in F$. Put simply, the color of the word $s_1 s_2 \dots s_k$ is the (original) color of the number $s_1 + s_2 + \cdots + s_k$.

By the Hales–Jewett Theorem, there is a word $w(x) = w_1 w_2 \dots w_k$ over $\{1, \dots, m, x\}$ in which the variable x appears, say, $d \geq 1$ times, and such that the words $w(1), \dots, w(m)$ are of the same color. For each $j = 1, \dots, m$, the color of $w(j)$ is the initial color of

$$\sum_{\substack{i=1 \\ w_i \neq x}}^{k} w_i + \sum_{\substack{i=1 \\ w_i = x}}^{k} j = a + jd,$$

where a is the sum of the constant (non-variable) letters in the word $w(x)$. Thus, the arithmetic progression $a + d, a + 2d, \dots, a + md$ is monochromatic.

Exercise 2.1. Let $d_1, d_2, \dots$ be a sequence of natural numbers. Prove that for each finite coloring of $\mathbb{N}$ and each m, there are there is a monochromatic arithmetic progression $a, a + d, \dots, a + (m-1)d$ with $d \in \mathrm{FS}(d_1, d_2, \dots)$.
Hint: As in the previous argument, define $\chi(s_1 s_2 \dots s_k) := c(d_1 s_1 + d_2 s_2 + \cdots + d_k s_k)$.

A small variation of the same idea provides the following result of Gallai.

DEFINITION 2.2. Let n be a natural number, and let $F \subseteq \mathbb{N}^n$. A *homothetic copy* of F is a set of the form $\mathbf{a} + dF$ with $\mathbf{a} \in \mathbb{N}^n \cup \{\mathbf{0}\}$ and $d \in \mathbb{N}$.

In other words, a homothetic copy is a copy of the given set, up to blowing by multiplying by a constant and shifting, without any distortion. The following theorem asserts that for each finite coloring of the "discrete n-dimensional space" $\mathbb{N}^n$, one can find in $\mathbb{N}^n$ a monochromatic homothetic copy of any desired finite pattern. The case of coloring the discrete plane ($n = 2$) is visually most appealing.

THEOREM 2.3 (Gallai). *Let n be a natural number, and let $F \subseteq \mathbb{N}^n$ be a finite set. For each finite coloring of the set $\mathbb{N}^n$, there are an element $\mathbf{a} \in \mathbb{N}^n \cup \{\mathbf{0}\}$ and a natural number d such that the set $\mathbf{a} + dF$ is monochromatic.*

PROOF. Let Σ be the free semigroup over the given finite set F. Define a coloring of Σ as follows: For all $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in F$, the color of the word $\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_k$ is the given color of the element

$$\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$$

of $\mathbb{N}^n$. By the Hales–Jewett Theorem, there are elements $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k \in F \cup \{x\}$ such that $\mathbf{w}_i = x$ for $d \geq 1$ values of i, and such that for

$$w(x) := \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_k,$$

the set $\{\, w(\mathbf{u}) : \mathbf{u} \in F \,\}$ is monochromatic. For each $\mathbf{u} \in F$, the color of the element $w(\mathbf{u})$ is the given color of

$$\sum_{\substack{i=1 \\ w_i \neq x}}^{k} \mathbf{w}_i + \sum_{\substack{i=1 \\ w_i = x}}^{k} \mathbf{u} = \mathbf{a} + d\mathbf{u},$$

where $\mathbf{a}$ is the sum of the non-variable elements $\mathbf{w}_i$. If $\mathbf{w}_i = x$ for all i, define $\mathbf{a} := \mathbf{0}$. Then the set $\mathbf{a} + dF = \{\, \mathbf{a} + d\mathbf{u} : \mathbf{u} \in F \,\}$ is monochromatic. □

EXERCISE 2.4. Prove that, in Gallai's Theorem, we may request that $d \in \mathrm{FS}(d_1, d_2, \dots)$, for any prescribed elements $d_1, d_2, \dots \in \mathbb{N}$ (see Exercise 2.1).

One may view Gallai's Theorem as a multi-dimensional version of van der Waerden's Theorem: for $n = 1$, the following corollary reproduces van der Waerden's Theorem.

COROLLARY 2.5 (Gallai). *Let n be a natural number. For each finite coloring of the set $\mathbb{N}^n$ and each m, there are elements $a_1, a_2, \dots, a_n \in \mathbb{N} \cup \{0\}$ and a natural number d such that, for the arithmetic progressions*

$$\begin{aligned}
A_1 &= \{a_1 + d, a_1 + 2d, \dots, a_1 + md\} \\
A_2 &= \{a_2 + d, a_2 + 2d, \dots, a_2 + md\} \\
&\vdots \\
A_n &= \{a_n + d, a_n + 2d, \dots, a_n + md\},
\end{aligned}$$

the set $A_1 \times \cdots \times A_n$ is monochromatic.

PROOF. Take $F = \{1, \dots, m\}^n$ and apply Theorem 2.3. For

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

we have that $A_1 \times \cdots \times A_n = \mathbf{a} + d \cdot \{1, \dots, m\}^n$. □

Thus, Corollary 2.5 is an immediate consequence of Theorem 2.3. On the other hand, as for each finite set $F \subseteq \mathbb{N}^n$ there is m with $F \subseteq \{1, \dots, m\}^n$, we also have that Theorem 2.3 is an immediate consequence of Corollary 2.5.

EXERCISE 2.6. Prove Gallai's Theorem, with the stronger requirement that $\mathbf{a} \in \mathbb{N}^n$. *Hint*: In the definition of the coloring, add a constant vector to the sum.

EXERCISE 2.7. Using the Compactness Theorem, formulate and prove a finite version of Gallai's Theorem. Formulate

3. Monochromatic affine subspaces

We have seen in Theorem 4.1.8 that, for each infinite-dimensional vector space V over the two-element field $\mathbb{Z}_2$ and each finite coloring of V, there is a monochromatic infinite-dimensional affine subspace of V. We have also seen in Exercise 4.1.10 that the same assertion for vector spaces over other fields fails. Relaxing "infinite-dimensional" to arbitrarily large finite dimension, we obtain the following theorem.

THEOREM 3.1. *Let V be an infinite-dimensional vector space over a finite field $\mathbb{F}$. For each finite coloring of V, there are monochromatic affine subspaces of V of arbitrarily large finite dimensions.*

PROOF. We will find a 2-dimensional monochromatic affine subspace. The proof for larger dimensions is similar. Fix linearly independent vectors $v_1, u_1, v_2, u_2, \dots \in V$. Let Σ be the free semigroup over the alphabet set $\mathbb{F}^2$. Given a finite coloring of V, define a coloring of Σ as follows: the color of the word $\binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_k}{\beta_k}$ is the given color of the vector

$$\alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_k v_k.$$

By the Hales–Jewett Theorem, there is a word $w(x) = \mathbf{w}_1 \cdots \mathbf{w}_k$ over $\mathbb{F}^2 \cup \{x\}$, where the variable x appears at least once, such that the words $w(\binom{\alpha}{\beta})$, for $\alpha, \beta \in \mathbb{F}$, have the same color. The color of the word $w(\binom{\alpha}{\beta})$ is the given color of the vector

$$\underbrace{\sum_{\substack{i=1\\ \mathbf{w}_i \neq x}}^{k} \alpha_i v_i + \beta_i u_i}_{v_0} + \sum_{\substack{i=1\\ \mathbf{w}_i = x}}^{k} \alpha v_i + \beta u_i = v_0 + \alpha \underbrace{\sum_{\substack{i=1\\ w_i = x}}^{k}}_{v} v_i + \beta \underbrace{\sum_{\substack{i=1\\ w_i = x}}^{k}}_{u} u_i = v_0 + \alpha v + \beta u.$$

Thus, the affine subspace $v_0 + \operatorname{span}\{v, u\}$ is monochromatic. Since the vectors v and u are linearly independent, the dimension of this affine space is 2. $\square$

4. High-dimensional tic-tac-toe

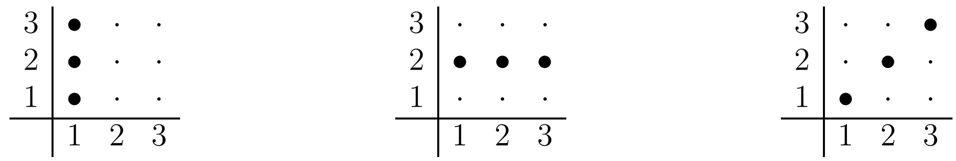
DEFINITION 4.1. A *line* in the n-dimensional discrete cube $\{1, \dots, m\}^n$ is a set of the form $\{w(1), \dots, w(m)\}$, where $w(x) \in \{1, \dots, m, x\}^n \setminus \{1, \dots, m\}^n$.

Figure 1 includes several examples of lines in the 2-dimensional cube (or, rather, square) $\{1, 2, 3\}^2$. The reader is encouraged to draw, mentally, the line xxx in the cube lines in $\{1, 2, 3\}^3$.

THEOREM 4.2. *Let k and m be natural numbers. For all large enough n, for each k-coloring of the set $\{1, \dots, m\}^n$ there is a monochromatic line.*

PROOF. Let Σ be the free semigroup over the alphabet set $\{1, \dots, m\}$. Let

$$\mathcal{A} = \{\, \{w(1), \dots, w(m)\} : w(x) \in \Sigma_x \setminus \Sigma \,\}.$$



(A) $w(x) = 1x$ (B) $w(x) = x2$ (C) $w(x) = xx$

FIGURE 1. Lines in the discrete square $\{1, 2, 3\}^2$

By the Hales–Jewett Theorem, for each k-coloring of Σ there is a monochromatic set in $\mathcal{A}$. By the Compactness Theorem, there is a finite set $H \subseteq \Sigma$ such that for each k-coloring of H there is a monochromatic set in $\mathcal{A}$. Let n be an upper bound on the length of the words in H.

Consider the set $\{1, \dots, m\}^{\leq n}$ of words of length up to n over $\{1, \dots, m\}$. Define a function

$$\begin{aligned} \varphi\colon \{1, \dots, m\}^{\leq n} &\to \{1, \dots, m\}^n \\ s_1 \dots s_r &\mapsto s_1 \dots s_r \underbrace{1 \dots 1}_{n-r}. \end{aligned}$$

The function φ maps every element of H into a line in $\{1, \dots, m\}^n$.

Let c be a k-coloring of the set $\{1, \dots, m\}^n$. Then $c \circ \varphi$ is a k-coloring of the set $\{1, \dots, m\}^{\leq n}$. In particular, this is a k-coloring of the set H. Let $\{w(1), \dots, w(m)\} \in H$ be monochromatic for this coloring. Then the line $\{\varphi(w(1)), \dots, \varphi(w(m))\}$ in $\{1, \dots, m\}^n$ is monochromatic for the coloring c. $\square$

Children loose interest in the game tic-tac-toe after little practice, discovering that every properly played play is a tie.

Let m and n be natural numbers. In the game *n-dimensional tic-tac-toe on a board of side m*, the first player chooses a position on the "board" $\{1, \dots, m\}^n$ and marks it X. The second player chooses an unoccupied position and marks it O, etc., until one of the players completes a line. The first player to complete a line wins (see Figure 2).

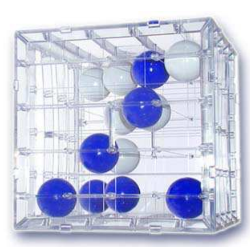


FIGURE 2. A typical configuration in a 3-dimensional tic-tac-toe game

As in the ordinary, 2-dimensional tic-tac-toe game, it may well be that all positions are occupied and there is no winner. By Theorem 4.2 for $k = 2$ colors, for each m there is n such that, in boards of dimensions n or larger, one of the players necessarily wins.

Consider a 2-player generalization of tic-tac-toe, where the board is $\{1, \dots, m\}^n$. By Theorem 4.2, if m is fixed and n is large enough, then in each play there is a winner. We will see, by a strategy-stealing argument, that this must be the first player.

THEOREM 4.3 (Zorn). *Consider a 2-player game where each player, in turn, puts a piece on a finite, completely visible board. Then either the first player has a winning strategy, or else the second player can force a tie or win.*

PROOF. The theorem follows from De Morgan's Laws. For each i, we denote the possible legal moves in the ith step of the game by x_i. Let N be the number of available positions on the board, and assume, for convenience, that N is even. In the case where N is odd, one more quantifier is needed below. We let the game continue for the entire N steps even if at some step it is clear that some player wins. That is, a player wins a play whose steps are $(x_1, x_2, \ldots, x_N)$ if it is the winner of the play $(x_1, \ldots, x_n)$ for the first n where the play $(x_1, \ldots, x_n)$ has a winner.

Let the players be Alice and Bob, where Alice plays first. Having a winning strategy for Alice means that

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \exists x_{N-1} \forall x_N,\ (x_1, \ldots, x_N) \text{ is a win for Alice.}$$

Thus, the nonexitence of a winning stratgy for Alice means that

$$\forall x_1 \exists x_2 \forall x_3 \exists x_4 \cdots \forall x_{N-1} \exists x_N,\ (x_1, \ldots, x_N) \text{ is a tie, or a win for Bob,}$$

that is, Bob has a strategy forcing a win for Bob or a tie. □

Since for large enough n a tie is impossible in the n-dimensional tic-tac-toe, one of the players must have a winning strategy.

COROLLARY 4.4. *Let m be a natural number. For all large enough n, the first player has a winning strategy in the n-dimensional tic-tac-toe game on a board of side m.*

PROOF. We have seen that either Alice or Bob has a winnig strategy. Assume, towards a contradiction, that Bob has a winning strategy. Then Alice can use Bob's strategy, as follows. In the first step, Alice puts X in an arbitrary position. After each move of Bob, Alice erases her arbitrarily-positioned X and applies Bob's strategy to the resulting board configuration (after interchanging the names of the Xs and the Os on the board). She thus obtains a position for her next X. If this position is already occupied by her aribtrarily positioned X, she chooses an arbitrary free position.

In each step, the board looks as after applying Bob's strategy (with X and O interchanged) and adding some extra X. Since the strategy is winning for Bob, at some stage there will be a line of Xs, even without the arbitrarily positioned X. Thus, Alice also has a winning strategy; a contradiction. □

5. Comments for Chapter 6

Theorem 1.2 is proved in Alfred W. Hales and Robert I. Jewett, *Regularity and positional games*, Transactions of the American Mathematical Society, 1963.

Gallai's Theorem (Theorem 2.3) is due to Tibor Gallai (born Grünwald). It was first published in Richard Rado, *Note on combinatorial analysis*, Proceedings of the London Mathematical Society, 1943.

Theorems 4.2 and 4.2 are proved in Graham, Leeb and Rothschild, *Ramsey's Theroem for a class of categories*, Advances in Mathematics, 1972.

It is known that in every 2-coloring of the cube $\{1, 2, 3\}^3$ there is a monochromatic line. It follows that every tic-tac-toe play on this board is a win for one of the players and, by Zorn's Theorem, there is a winning strategy for Alice. Here is a simple winning strategy for Alice: First, occupy the central cell. Once Bob occupies his cell, choose a plane through the central cell that does not include Bob's cell, and henceforth play in this plane. In this plane, the central cell is marked X, all other cells are empty, and it is Alice's turn. Occupying the top horisontal line, for example, would force Bob to put his pieces on the bottom horisonal line. Alice will complete her line before Bob does. Interestingly, it is known that if Bob occupies the central cell in his first move, then *Bob* can force a win.

Consider the variation of tic-tac-toe where the first to complete a line *looses*. This is often refered to as the *misére* version of the game. In this case, Alice has a simple winning-or-tie strategy, in every board of odd side-length: Occupy the central cell in the first move. (In light of the previous paragraph, this is counter-intuitive. Be patient.) Then respond to each move of Bob by occupying the cell symmetrically opposite, with respect to the central cell, to the one occupied by Bob. If, at some point, a line is completed, but if this is done by Alice then there was earlier a symmetrically-positioned line completed by Bob.

CHAPTER 7

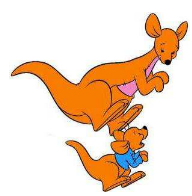
Monochromatic solutions for a linear equation

1. Piecewise syndetic sets

We have seen that a set of natural numbers is in an idempotent of $(\beta\mathbb{N},+)$ if and only if it is an FS set. In the proof of van der Waerden's Theorem, sets belonging to minimal elements of $(\beta\mathbb{N},+)$ were prominent. These sets can be defined combinatorially. For nonnegative integers n and $m \geq n$, define the interval $[n,m] := \{n, n+1, n+2, \dots, m\}$ of nonnegative integers.

Definition 1.1. An infinite set $A \subseteq \mathbb{N}$ is *syndetic* (from Greek: bound together) if the distances among consecutive elements in A are bounded by some constant. A *piecewise syndetic* set is an intersection of a syndetic set and a set containing arbitrarily long intervals of natural numbers.

One may think of syndetic sets as the landing points of a kangaroo traveling over the natural numbers. The distance among consecutive landing points is bounded by the maximum jumping distance of that kangaroo.



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 . . .

For sets $A, B \subseteq \mathbb{N}$ and a nonnegative integer n, define

$$\begin{aligned} A-n &:= \{\, a-n : a \in A \,\} \cap \mathbb{N} \\ A-B &:= \{\, a-b : a \in A,\, b \in B \,\} \cap \mathbb{N} \end{aligned}$$

Then:

(1) A set $A \subseteq \mathbb{N}$ is syndetic if and only if there is a natural number c such that $A-[0,c] = \mathbb{N}$.
(2) A set $A \subseteq \mathbb{N}$ is piecewise syndetic if and only if there is a natural number c such that the set $A-[0,c]$ contains arbitrarily long intervals of natural numbers.

Theorem 1.2. *Let $A \subseteq \mathbb{N}$. The following assertions are equivalent:*

(1) *A is a member of a minimal element of $(\beta\mathbb{N},+)$.*
(2) *A is piecewise syndetic.*

Proof. (1) $\Rightarrow$ (2): Assume that A belongs to a minimal ultrafilter p. Let $q \in \beta\mathbb{N}+p$. By the Fixing Lemma, there is $r \in \beta\mathbb{N}$ with $p = r+q$. Then $A \in r+q$, and thus there are n and $C \in q$ with $n+C \subseteq A$. Thus, $A-n \in q$. It follows that $\beta\mathbb{N}+p \subseteq \bigcup_n [A-n]$. By compactness of the left ideal $\beta\mathbb{N}+p$, there is a natural number c such that

$$\beta\mathbb{N}+p \subseteq [A] \cup [A-1] \cup [A-2] \cup \cdots \cup [A-c] = \big[A-[0,c]\big].$$

Let $A' = A-[0,c]$. We will show that the set A' contains arbitrarily long intervals. Let n be a natural number. As $n+p \in \beta\mathbb{N}+p \subseteq [A']$, we have that $n+p \in [A']$, that is, $A' \in n+p$.

Thus, there is $B_n \in p$ with $n + B_n \subseteq A'$. For each n, we have that $B_1 \cap \cdots \cap B_n \in p$. Fix an element $b \in B_1 \cap \cdots \cap B_n$. Then $n + b \in n + B_n \subseteq A'$, and therefore $b + 1, \ldots, b + n \in A'$.

(2) $\Rightarrow$ (1): Let c be a natural number such that the set $A' := A - [0, c]$ contains arbitrarily long intervals. For each n, as the set A' contains an interval of length greater than n, we have that

$$A' \cap (A' - 1) \cap \cdots \cap (A' - n) \neq \emptyset.$$

By the finite intersection property, the set $L = \bigcap_{n \geq 0}[A' - n]$ is nonempty. It is easy to verify that L is a left ideal of $\beta\mathbb{N}$.

Let $p \in L$ be a minimal element. Then

$$A \cup (A - 1) \cup \cdots \cup (A - c) = A' \in p,$$

and thus there is $i \leq c$ such that $A - i \in p$. Thus, $A \in i + p$. As p is minimal, so is the element $i + p$. $\square$

Exercise 1.3. Complete the proof of Theorem 1.2, by proving the following assertion: Let $A \subseteq \mathbb{N}$ be such that $A \cap (A-1) \cap \cdots \cap (A-n) \neq \emptyset$ for all n. Prove that the set $L = \bigcap_{n \geq 0}[A - n]$ is a left ideal of $\beta\mathbb{N}$.

The following definition extends the earlier definitions to arbitrary semigroups.

Definition 1.4. Let $(S, \cdot)$ be a semigroup and $A \subseteq S$.

(1) For an element $b \in S$, define $b^{-1}A := \{\, s \in S : bs \in A \,\}$.
(2) For a set $B \subseteq S$, define $B^{-1}A := \bigcup_{b \in B} b^{-1}A$.
(3) The set A is *syndetic* if there is a finite set $F \subseteq S$ such that $F^{-1}A = S$.
(4) The set A is *piecewise syndetic* if there is a finite set $F \subseteq S$ such that, for all n and all $s_1, \ldots, s_n \in S$, there is $x \in S$ with

$$\{s_1, s_2, \ldots, s_n\} \cdot x \subseteq F^{-1}A.$$

Exercise 1.5. Show that the original and general definition of piecewise syndetic sets in the semigroup $(\mathbb{N}, +)$ are equivalent.

By modifying the proof of Theorem 1.2 in accordance with the general definition, we have the following theorem.

Theorem 1.6. *Let S be a semigroup and $A \subseteq S$. The following assertions are equivalent:*

(1) *The set A is a member of a minimal element of βS.*
(2) *The set A is piecewise syndetic.*

Exercise 1.7. Prove Theorem 1.6.

Exercise 1.8. Let $(S, +)$ be an abelian semigroup and T be a subsemigroup of S. Let φ be the continuous extension of the identity embedding $\mathrm{id}\colon T \to \beta S$ defined by $\mathrm{id}(t) = t$ for all $t \in T$. Prove the following assertions:

(1) The set $\overline{T} = [T]$ is a subsemigroup of βS.
(2) The function $\varphi\colon \beta T \to [T]$ is an isomorphism of topological semigroups (a continuous bijective homomorphism).
(3) The function $f(p) := \{\, B : \exists A \in p,\ A \subseteq B \,\}$ is a continuous extension of the identity map on T, and thus $\varphi(p) = f(p)$ for all $p \in \beta T$.
(4) An element $p \in \beta T$ is minimal if and only if $\varphi(p)$ is minimal in $[T]$.
(5) If T is a left ideal of S, then $\varphi(p)$ is minimal in βS for all minimal $p \in \beta T$.

Exercise 1.9. Let $(S, +)$ be an abelian semigroup. Using the previous exercise, show that for each left ideal L of S:

(1) A set $A \subseteq L$ belongs to a minimal element in βL if and only if A belongs to a minimal element in βS.
(2) A set $A \subseteq S$ belongs to a minimal element in βS if and only if the set $A \cap L$ belongs to a minimal element in βL.

2. The Piecewise Syndetic Sets Theorem

Definition 2.1. Let $(S, +)$ be an abelian semigroup. If S has no neutral element 0, we fix an element $0 \notin S$ and define $s + 0 = 0 + s = s$ for all $s \in S$.

Let $(s_1, s_2, \dots)$ be a sequence in S. For a nonempty finite set $F \subseteq \mathbb{N}$ we define $s_F := \sum_{n\in F} s_n$. For the empty set $F = \emptyset$, define $s_\emptyset := 0$.

The notation s_F generalizes in a natural manner the notation s_n for the n-th element of a sequence. Its basic properties include the following ones:

(1) $s_n = s_{\{n\}}$.
(2) $\mathrm{FS}(s_1, s_2, \dots) = \{\, s_F : \emptyset \neq F \in [\mathbb{N}]^{<\infty} \,\}$.
(3) If $F_1 < F_2$ (or just $F_1 \cap F_2 = \emptyset$) then $s_{F_1} + s_{F_2} = s_{F_1\cup F_2}$.

Let $(S, \cdot)$ be a semigroup. For elements $a, d_1, \dots, d_m \in S$, it is customary to define

$$a \cdot \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} := \begin{pmatrix} ad_1 \\ \vdots \\ ad_m \end{pmatrix}.$$

The following definition follows this convention.

Definition 2.2. Let $(S, +)$ be an additive semigroup. For elements $a, d_1, \dots, d_m \in S$, define

$$a + \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} := \begin{pmatrix} a + d_1 \\ \vdots \\ a + d_m \end{pmatrix}.$$

For an element $a \in S$ and m fixed in the background, we define

$$\vec{a} := \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \in S^m.$$

Thus, for an element $a \in S$ and a vector $\mathbf{v} \in (S \cup \{0\})^m$, we have that $a + \mathbf{v} = \vec{a} + \mathbf{v}$.

For a finite set $F \subseteq \mathbb{N}$ and a natural number n, we write $n < F$ if all elements of F are greater than n. We define $F \leq n$ and $F < n$ in a similar manner.

Theorem 2.3 (Piecewise Syndetic Sets). *Let $(S, +)$ be an abelian semigroup, $A \subseteq S$ be a piecewise syndetic set and m be a natural number. For all $\mathbf{v}_1, \mathbf{v}_2, \dots \in (S \cup \{0\})^m$, there are a finite nonempty set $F \subseteq \mathbb{N}$ and an element $a \in A$ such that*

$$a + \mathbf{v}_F = a + \sum_{n\in F} \mathbf{v}_n \in A^m.$$

Proof. This proof combines the proof that every FS set belongs to an idempotent with the proof of van der Waerden's Theorem (or the Hales–Jewett Theorem).

We work in the semigroup S^m. In accordance with our notation, $\mathbf{v}_\emptyset := \vec{0}$. Notice that $a + \mathbf{v} \in S^m$ for all $a \in S$ and $\mathbf{v} \in (S \cup \{0\})^m$. For each n, let

$$T_n = \{\, a + \mathbf{v}_F : a \in S,\ n < F \in [\mathbb{N}]^{<\infty} \,\}.$$

Let $T = \bigcap_n \overline{T_n} \subseteq (\beta S)^m$. For each n, we have that $\{\, \vec{a} : a \in S \,\} \subseteq T_n$, and thus $\{\, \vec{a} : a \in S \,\} \subseteq T$. In particular, $T \neq \emptyset$.

The set T is a subcompany of $(\beta S)^m$: Let $\mathbf{x}, \mathbf{y} \in T$. We need to show that $\mathbf{x} + \mathbf{y} \in \overline{T_n}$ for all n. Fix a natural number n. Let U be a neighborhood of $\mathbf{x} + \mathbf{y}$. By right continuity, there is a neighborhood V of $\mathbf{x}$ such that $V + \mathbf{y} \subseteq U$. As $\mathbf{x} \in \overline{T_n}$ and V is a neighborhood of $\mathbf{x}$, there is an element $a + \mathbf{v}_F \in V \cap T_n$. Since $(a + \mathbf{v}_F) + \mathbf{y} \in U$ and the left element of the sum is in S^m, there is a neighborhood W of $\mathbf{y}$ such that $(a + \mathbf{v}_F) + W \subseteq U$. Fix a number $n_1 > F$. Take an element $b + \mathbf{v}_H \in W \cap T_{n_1}$. As $n < F < H$ and the semigroup S is commutative,

$$a + \mathbf{v}_F + b + \mathbf{v}_H = a + b + \mathbf{v}_{F \cup H} \in T_n \cap U.$$

In particular, the intersection $T_n \cap U$ is nonempty.

For each n, let

$$I_n = \{\, a + \mathbf{v}_F : a \in S,\ n < F \in [\mathbb{N}]^{<\infty},\ F \neq \emptyset \,\}.$$

Let $I = \bigcap_n \overline{I_n}$. Then $\overline{I_1} \supseteq \overline{I_2} \supseteq \cdots$ and $\overline{I_n} \supseteq I_n \neq \emptyset$ for all n. By the finite intersection property, the set I is nonempty.

Notice that, in the above proof that T is a semigroup, if either F or H is nonempty, then the sum is in $I_n \cap U$. Thus, I is an ideal of the company T.

Let $p \in \beta S$ be a minimal element such that $A \in p$. As $\{\, \vec{a} : a \in S \,\} \subseteq T$ and T is a closed set, the vector

$$\mathbf{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix}$$

is in T.

By Lemma 1.8, the vector $\mathbf{p}$ is minimal in T and is thus in I. As the set $[A]^m$ is a neighborhood of $\mathbf{p}$, it intersects the set I_1 (for example), and thus there is an element $a + \mathbf{v}_F \in A^m$.

To guarantee that $a \in A$, take the vectors

$$\mathbf{u}_1 := \begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}, \mathbf{u}_2 := \begin{pmatrix} 0 \\ \mathbf{v}_2 \end{pmatrix}, \dots \in (S \cup \{0\})^{m+1}.$$

We have proved that there are a nonempty finite set $F \subseteq \mathbb{N}$ and an element $a \in S$ such that $a + \mathbf{u}_F \in A^{m+1}$. Then

$$a + \mathbf{u}_F = a + \begin{pmatrix} 0 \\ \mathbf{v}_F \end{pmatrix} = \begin{pmatrix} a \\ a + \mathbf{v}_F \end{pmatrix},$$

and we have that $a \in A$ and $a + \mathbf{v}_F \in A^m$. $\square$

EXERCISE 2.4. In the notation of the last proof, prove that the set I is indeed an ideal of the company T.

EXERCISE 2.5. Let $(S, +)$ be an infinite abelian semigroup with a neutral element 0. Prove the following assertions:

(1) The set $\{0\}$ is not piecewise syndetic in S.
(2) In the Piecewise Syndetic Sets Theorem for S, we may request that all coordinates of the vector $a + \mathbf{v}_F$ are nonzero.

Hint: For (1), if $t + s + x = 0$ then s is an inverse of $t + x$.

The Piecewise Syndetic Sets Theorem easily implies van der Waerden's Theorem: For example, to find a monochromatic arithmetic progression of length 5, take

$$\mathbf{v}_n = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

for all n. Given a finite coloring of $\mathbb{N}$, let $A \subseteq \mathbb{N}$ be a monochromatic piecewise syndetic set. Such a set exists since in every ultrafilter there is a monochromatic set. By the Piecewise Syndetic Sets Theorem, there are a natural number a and a finite nonempty set $F \subseteq \mathbb{N}$ such that

$$a + \mathbf{v}_F = \begin{pmatrix} a \\ a \\ a \\ a \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ a+d \\ a+2d \\ a+3d \\ a+4d \end{pmatrix} \in A^5,$$

where $d = |F|$. We have thus found a monochromatic arithmetic progression of length 5.

The following strong version of van der Waerden's Theorem will be used below for finding monochromatic solutions of equations.

THEOREM 2.6. *Let $m, d_1, d_2, \dots$ be natural numbers. For each piecewise syndetic set A, there are an element $d \in \mathrm{FS}(d_1, d_2, \dots)$ and a natural number a such that $a, a+d, \dots, a+(m-1)d \in A$.*

PROOF. The proof is similar to the derivation of van der Waerden's Theorem from the Piecewise Syndetic Sets Theorem. For each n, take

$$\mathbf{v}_n := \begin{pmatrix} 0 \\ d_n \\ 2d_n \\ \vdots \\ (m-1)d_n \end{pmatrix}.$$

Then

$$a + \mathbf{v}_F = a + \sum_{n \in F} \begin{pmatrix} 0 \\ d_n \\ \vdots \\ (m-1)d_n \end{pmatrix} = a + \begin{pmatrix} 0 \\ \sum_{n \in F} d_n \\ \vdots \\ (m-1)\sum_{n \in F} d_n \end{pmatrix} = \begin{pmatrix} a \\ a+d \\ \vdots \\ a+(m-1)d \end{pmatrix},$$

where $d = \sum_{n \in F} d_n$. □

3. Rado's Theorem

Schur's Coloring Theorem asserts that the equation $x + y - z = 0$ has a monochromatic solution for each finite coloring of $\mathbb{N}$. In this section, we identify all homogeneous linear equations with this property.

LEMMA 3.1. *Let e be an idempotent element of the semigroup $(\beta\mathbb{N}, +)$. Then $c\mathbb{N} \in e$ for all natural numbers c.*

Proof. This is Exercise 3.6.7. We recast that proof in a slightly different manner. Since

$$c\mathbb{N} \cup (c\mathbb{N}+1) \cup \dots \cup (c\mathbb{N}+c-1) = \mathbb{N} \in e,$$

there is $i \in \{0, \dots, c-1\}$ such that $c\mathbb{N}+i \in e$. As e is an idempotent, the set $c\mathbb{N}+i$ is an FS set. In particular, there are $cm+i, cn+i \in c\mathbb{N}+i$ whose sum $c(m+n)+2i$ is also in $c\mathbb{N}+i$. Thus, $i \in c\mathbb{N}$, and therefore $i = 0$. $\square$

Exercise 3.2. Prove that, for each element $p \in \beta\mathbb{N}$, the number $p \bmod c$ is the unique k such that $c\mathbb{N}+k \in p$.

To realize that the following theorem is a nontrivial upgrade of van der Waerden's Theorem, notice that, while the latter theorem is trivial for arithmetic progressions of length two, the following theorem is not.

Recall that a *minimal idempotent* in a semigroup is a minimal element that is also an idempotent. Minimal idempotents exist in every company, since minimal left ideals are subcompanies.

Theorem 3.3. *For each finite coloring of $\mathbb{N}$ and all natural numbers m and c there are natural numbers a and d such that the terms in the arithmetic progression $a, a+d, \dots, a+(m-1)d$ and the product cd are all of the same color.*

Proof. Let e be a minimal idempotent in $\beta\mathbb{N}$. Take a monochromatic set $A \in e$. As e is an idempotent, $c\mathbb{N} \in e$ and thus $A \cap c\mathbb{N} \in e$. Using again that e is an idempotent, the set $A \cap c\mathbb{N}$ is an FS set. Let $cd_1, cd_2, \dots \in A \cap c\mathbb{N}$ be distinct elements such that

$$c \cdot \mathrm{FS}(d_1, d_2, \dots) = \mathrm{FS}(cd_1, cd_2, \dots) \subseteq A \cap c\mathbb{N}.$$

By Theorem 2.6, since the set A is piecewise syndetic, there is an element $d \in \mathrm{FS}(d_1, d_2, \dots)$ such that $a, a+d, \dots, a+(m-1)d \in A$. By the above equation, we have that $cd \in A$. $\square$

Theorem 3.4 (Rado). *For a homogeneous linear equation*

$$a_1x_1 + \dots + a_mx_m = 0$$

with nonzero integer coefficients, the following properties are equivalent:

(1) *Some of the coefficients $a_1, \dots, a_m$ sum to 0.*
(2) *For each finite coloring of $\mathbb{N}$, the given equation has a monochromatic solution.*

Proof. (1) $\Rightarrow$ (2): If $a_1 + \dots + a_m = 0$, then the vector $(1, \dots, 1)$ is a monochromatic solution. Thus, assume that $a_1 + \dots + a_m \neq 0$. By reordering the equation, we may assume that $a_1 + \dots + a_k = 0$ for some maximal $k < m$. Since the coefficients are nonzero, k is greater than 1.

Consider first the simple case, where the equation is $cx - cy + bz = 0$, for nonzero integers c and b. Multiplying the given equation by -1, if needed, we may assume that $b > 0$. By exchanging the roles of x and y, if needed, we may also assume that $c > 0$. We request that

$$y - x = \frac{b}{c}z.$$

For the solution to be a natural number, we may take $z = cd$ for a natural number d. Then $y = x + bd$, that is, $(x, x+bd, cd)$ is a solution for all $d \in \mathbb{N}$. By Theorem 3.3 with $m > b$, there is such a monochromatic triple.

We now show that the general case follows from the above simple case. Let $c := a_1+\dots+a_{k-1}$ and $b := a_{k+1} + \dots + a_m$. Then $c + a_k = 0$, and thus $c = -a_k \neq 0$. Since $a_1 + \dots + a_k + b \neq 0$ and $a_1 + \dots + a_k = 0$, we have that $\beta \neq 0$. By the simple case treated above, the equation

$cx - cy + bz = 0$ has a monochromatic solution (x, y, z) in $\mathbb{N}$. Then $(x, \dots, x, y, z, \dots, z)$ is a monochromatic solution of the original equation:

$$\begin{aligned} &a_1x + \dots + a_{k-1}x + a_ky + a_{k+1}z + \dots + a_mz = \\ &= \underbrace{(a_1 + \dots + a_{k-1})}_{c}x + \underbrace{a_k}_{-c}y + \underbrace{(a_{k+1} + \dots + a_m)}_{b}z = 0. \end{aligned}$$

(2) $\Rightarrow$ (1): Fix a very large prime number p (it suffices to have $|a_1| + \dots + |a_m| < p$.) Define a coloring $\chi\colon \mathbb{N} \to \{1, \dots, p-1\}$ by letting $\chi(n)$ be the maximal i such that $n = p^k(pt + i)$, for some natural number t, where p^k is the maximal power of p dividing n. By the assumption, there is a monochromatic solution $x_1 = p^{k_1}(pt_1 + i), \dots, x_m = p^{k_m}(pt_m + i)$. By reordering the coefficients of the equation, we may assume that, for some $j \in \{1, \dots, m\}$,

$$k_1 = \dots = k_j < k_{j+1} \leq \dots \leq k_m.$$

Since

$$a_1p^{k_1}(pt_1 + i) + a_2p^{k_2}(pt_1 + i) + \dots + a_mp^{k_m}(pt_m + i) = a_1x_1 + \dots + a_mx_m = 0,$$

We may divide the equation by p^{k_1} and reduce it modulo p, to obtain

$$(a_1 + \dots + a_j)i = a_1i + \dots + a_ji = 0 \pmod{p}.$$

Thus, $a_1 + \dots + a_j = 0 \bmod p$. Since $|a_1 + \dots + a_j| < p$, we have that $a_1 + \dots + a_j = 0$. $\square$

The given proof of Theorem 3.4 establishes the following result. Notice that a set $A \subseteq \mathbb{N}$ is a piecewise syndetic FS set if and only if it is a union of a piecewise syndetic set and an FS set. In particular, sets in minimal idempotents of $(\beta\mathbb{N}, +)$ are piecewise syndetic FS sets.

THEOREM 3.5. *Consider a homogeneous linear equation with nonzero integer coefficients,*

$$a_1x_1 + \dots + a_mx_m = 0,$$

with some of its coefficients summing up to 0. For each piecewise syndetic FS set $A \subseteq \mathbb{N}$, there is a solution $(x_1, \dots, x_m)$ with all coordinates in A. $\square$

EXERCISE 3.6. Prove that, for each finite coloring of $\mathbb{N}$, there is a color such that for each homogeneous linear equation with integer coefficients such that some of its coefficients sum to 0, there is a solution of that color.

4. Comments for Chapter 7

Theorem 2.3 is proved in Neil Hindman, Dibyendu De and Dona Strauss, *A new and stronger Central Sets Theorem*, Fundamenta Mathematicae, 2008.

Theorem 3.3 is due to Hillel Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

Richard Rado, a student of Schur, proved Theorem 3.4 in his paper *Studien zur Kombinatorik*, Mathematische Zeitschrift, 1933. In the proof of Rado's Theorem, one may wish to restrict to solutions where all variables take distinct values. This may be done, and will be treated later.

Theorem 3.5 may be new, but this is what the present proof of Rado's Theorem really gives. In this theorem, we cannot request that the variables take distinct values.

CHAPTER 8

Monochromatic images and solutions

1. The Central Sets Theorem

LEMMA 1.1. *Let $(S, +)$ be a semigroup, e be an idempotent of βS, $A \in e$, and $m \in \mathbb{N}$. There is a set $B \subseteq A$ in e such that, for each $\mathbf{v} \in B^m$, there is a set $C \subseteq A$ in e with $\mathbf{v} + C^m \subseteq A^m$.*

PROOF. By the Idempotent Characterization Theorem, there is a set $B \subseteq A$ in e such that, for each $b \in B$, there is a set $C_b \subseteq A$ in e with $b + C_b \subseteq A$. For each vector

$$\mathbf{v} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in C^m,$$

we have that $C := C_{b_1} \cap \cdots \cap C_{b_m} \in e$. Then $\mathbf{v} + C^m \subseteq A^m$. □

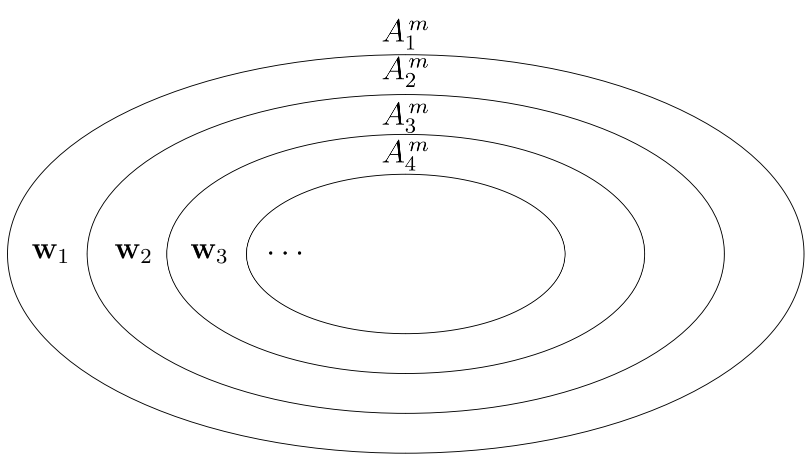
The following notion is stronger than being piecewise syndetic and FS.

DEFINITION 1.2. A set $A \subseteq \mathbb{N}$ is *central* if it is a member of a minimal idempotent of $(\beta\mathbb{N}, +)$.

THEOREM 1.3 (Central Sets). *Let $(S, +)$ be an abelian semigroup, $A \subseteq S$ be a central set and m be a natural number. For all $\mathbf{v}_1, \mathbf{v}_2, \dots \in (S \cup \{0\})^m$, there are nonempty finite sets of natural numbers $F_1 < F_2 < \cdots$ and elements $a_1, a_2, \dots \in A$ such that*

$$\mathrm{FS}(a_1 + \mathbf{v}_{F_1}, a_2 + \mathbf{v}_{F_2}, \dots) \subseteq A^m.$$

PROOF. Let e be a minimal idempotent of βS with $A \in e$. We proceed as in the proof of Hindman's Theorem.



We use Lemma 1.1 repeatedly. Let $A_1 := A$.

Choose an element $B \in e$ as in the lemma, for A_1. Let $\mathbf{w}_1 \in B^m$. By the lemma, there is a set $A_2 \subseteq A_1$ in e such that $\mathbf{w}_1 + A_2^m \subseteq A_1^m$.

Choose an element $B \in e$ as in the lemma, for A_2. Let $\mathbf{w}_2 \in B^m$. By the lemma, there is a set $A_3 \subseteq A_2$ in e such that $\mathbf{w}_2 + A_3^m \subseteq A_2^m$.

Continue in the same manner. It follows, as in the proof of Hindman's Theorem, that

$$\mathrm{FS}(\mathbf{w}_1, \mathbf{w}_2, \dots) \subseteq A^m.$$

In each step n of the construction, the vector $\mathbf{w}_n$ may be chosen to be any element in a power of a central set. Thus, by the Piecewise Syndetic Sets Theorem, we may request that the vector $\mathbf{w}_1$ is of the form $a_1 + \mathbf{v}_{F_1}$, where $a_1 \in B$. In particular, $a_1 \in A$.

By the Piecewise Syndetic Sets Theorem with the vectors $\{\, \mathbf{v}_n : n > F_1 \,\}$, we may request that the vector $\mathbf{w}_2$ is of the form $a_2 + \mathbf{v}_{F_2}$, where $F_1 < F_2$ and $a_2 \in B \subseteq A$.

Continuing in this manner, we see that we may request that

$$\mathbf{w}_n = a_n + \mathbf{v}_{F_n},$$

$F_n < F_{n+1}$ and $a_n \in A$ for all n. $\square$

EXERCISE 1.4. Prove that, in the Central Sets Theorem, we may request, in addition, that $\mathrm{FS}(a_1, a_2, \dots) \subseteq A$.
Hint: Consult the proof of the Piecewise Syndetic Sets Theorem.

COROLLARY 1.5. *Let $(S, +)$ be an abelian semigroup and m be a natural number. For each finite coloring of S and all $\mathbf{v}_1, \mathbf{v}_2, \dots \in (S \cup \{0\})^m$, there are a color, nonempty finite sets of natural numbers $F_1 < F_2 < \dots$, and elements $a_1, a_2, \dots \in S$, such that the coordinates of the vectors in the set $\mathrm{FS}(a_1 + \mathbf{v}_{F_1}, a_2 + \mathbf{v}_{F_2}, \dots)$ are all of that color.*

PROOF. Let e be a minimal idempotent of βS. Take a monochromatic set $A \in e$ and apply the Central Sets Theorem. $\square$

We will use below that the finite sums in the Central Sets Theorem are of the following form: For all $i_1 < i_2 < \dots < i_k$,

$$(a_{i_1} + \mathbf{v}_{F_{i_1}}) + \dots + (a_{i_k} + \mathbf{v}_{F_{i_k}}) = a_{i_1} + \dots + a_{i_k} + \mathbf{v}_{F_{i_1}} + \dots + \mathbf{v}_{F_{i_k}} = a + \mathbf{v}_F,$$

where $a = a_{i_1} + \dots + a_{i_k}$ and $F = F_{i_1} \cup \dots \cup F_{i_k}$. For a finite set $H \subseteq \mathbb{N}$, write $F_H := \bigcup_{n \in H} F_n$. Then

$$\sum_{n \in H} a_n + \mathbf{v}_{F_n} = a_H + \mathbf{v}_{F_H}.$$

2. Monochromatic images

DEFINITION 2.1. Let $\mathbf{A}$ be a matrix of nonnegative integers. An entry a_{ij} of the matrix $\mathbf{A}$ is *first* if it is the first nonzero entry in its row. A matrix $\mathbf{A}$ has the *first entries property* if it has no zero rows (so that each row as a first entry) and, in each column of $\mathbf{A}$, the first entries are equal.

In the definition of the first entries property, we do not request that there are first entries in every column of the matrix.

In this section, we will prove the following theorem.

THEOREM 2.2 (Monochromatic Image). *Let $\mathbf{A}$ be an $m \times n$ matrix of nonnegative integers with the first entries property. For each finite coloring of $\mathbb{N}$, there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that all coordinates of the vector $\mathbf{A}\mathbf{v}$ are of the same color.*

Moreover, for each central set $A \subseteq \mathbb{N}$ there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that $\mathbf{A}\mathbf{v} \in A^m$.

As usual, to see that the second part of the theorem implies the first, fix a minimal idempotent $e \in \beta\mathbb{N}$ and recall that, given a finite coloring of $\mathbb{N}$, there is in e a monochromatic set A. The set A is central. This provides a stronger assertion that, for each finite coloring of $\mathbb{N}$, there is a color such that all matrices with the first entries property have image vectors with all entries *of that color*.

Before proving this theorem, we illustrate it by drawing from it several earlier theorems. Notice that all matrices in the following three examples have the first entries property. Using that

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\\ x+y\end{pmatrix},$$

we obtain Schur's Coloring Theorem. Using that

$$\begin{pmatrix}1 & 0\\ 1 & 1\\ 1 & 2\\ \vdots & \vdots\\ 1 & m\\ 0 & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ x+y\\ x+2y\\ \vdots\\ x+my\\ cy\end{pmatrix},$$

we obtain the upgraded van der Waerden Theorem (Theorem 7.3.3). We can also obtain the finite version of Hindman's Theorem (Exercise 4.1.4). For example, to have three natural numbers and all their (finite) sums of the same color, we use that

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}x\\ y\\ z\\ x+y\\ x+z\\ y+z\\ x+y+z\end{pmatrix}.$$

Exercise 2.3. Prove, using the Monochromatic Image Theorem, that for all natural numbers m, c_1 and c_2, for each finite coloring of $\mathbb{N}$ there are natural numbers a and d such that

(1) c_1 divides a.
(2) The numbers $a, a+d, \ldots, a+md$ and $c_2 d$ have the same color.

Every matrix of the form

$$\begin{pmatrix}\vec{a}_1 & * & \cdots & *\\ \mathbf{0} & \vec{a}_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & *\\ \mathbf{0} & \cdots & \mathbf{0} & \vec{a}_n\end{pmatrix}, \tag{1}$$

where: $a_1, \ldots, a_n$ are natural numbers, the number of entries in each vector

$$\vec{a_i}=\begin{pmatrix}a_i\\ \vdots\\ a_i\end{pmatrix}$$

is unlimited, and the asterisk symbols "$*$" may be replaced by arbitrary vectors nonnegative integers, has the first entries property.

For the following reasons, it suffices to prove the Monochromatic Image Theorem for matrices of the form (1):

(1) If a certain column is the zero vector, then the corresponding entry in the vector $\mathbf{v}$ has no effect on the image vector $\mathbf{Av}$. Thus, we may assume that the matrix $\mathbf{A}$ has no zero columns.
(2) If we permute the order of the rows of the matrix $\mathbf{A}$, the entries of the image vector $\mathbf{Av}$ are just permuted accordingly.
(3) By adding rows to the matrix while preserving the first entries property, the claim in the theorem only becomes stronger: by the previous item, we may assume that the rows are added at the bottom of the matrix, and then the old image vector is an initial segment of the new one. Thus, we may assume that there are first entries in every column of the given matrix.

To see more clearly the connection of the following proof to the Central Sets Theorem, it is recommended to read it first under the assumption that $a_i = 1$ for all i in the matrix presentation (1).

Proof of the Monochromatic Image Theorem. We may assume that the matrix $\mathbf{A}$ is of the form (1).

Let C be a central set. We will find a vector $\mathbf{v} \in \mathbb{N}^n$ such that all entries of the image vector $\mathbf{Av}$ are in C. The proof is by induction on n. In order to carry out the induction step more easily, we will prove a stronger assertion: there are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \in \mathbb{N}^n$ such that, for each finite nonempty set $F \subseteq \mathbb{N}$, all entries of the vector $\mathbf{Av}_F$ are in C.

$n = 1$: In this case, the matrix is a vector with all entries identical. As rows identical to previous rows do not contribute a new entry to the image vector, we may assume that each row appears exactly once. In our case, this means that the matrix is a scalar, $a = a_1$, and we need to find scalars $\mathbf{v}_1, \mathbf{v}_2, \dots \in \mathbb{N}$ such that, for each nonempty finite set $F \subseteq \mathbb{N}$, $a\mathbf{v}_F \in C$. Since the sets C and $a\mathbb{N}$ belong to the same idempotent ultrafilter, the set $C \cap a\mathbb{N}$ is an FS set. Thus, there are elements $a\mathbf{v}_1, a\mathbf{v}_2, \dots \in C \cap a\mathbb{N}$ such that

$$a\,\mathrm{FS}(\mathbf{v}_1, \mathbf{v}_2, \dots) = \mathrm{FS}(a\mathbf{v}_1, a\mathbf{v}_2, \dots) \subseteq C \cap a\mathbb{N} \subseteq C.$$

$n + 1$: Represent the matrix (1) in the block form

$$\begin{pmatrix} \vec{a} & \mathbf{B} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}.$$

By duplicating rows, if needed, we may assume that the number of rows in the matrices $\mathbf{A}$ and $\mathbf{B}$ is equal, and denote it m.

The matrix $\mathbf{A}$ is of the form (1), with n columns. By the inductive hypothesis, there are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \in \mathbb{N}^n$ such that, for each nonempty finite set $F \subseteq \mathbb{N}$, all entries of the vector $\mathbf{Av}_F$ are in C. For each $b \in \mathbb{N}$ and all nonempty finite sets $F \subseteq \mathbb{N}$, we have that

$$\begin{pmatrix} \vec{a} & \mathbf{B} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \begin{pmatrix} b \\ \mathbf{v}_F \end{pmatrix} = \begin{pmatrix} ab + \mathbf{Bv}_F \\ \mathbf{Av}_F \end{pmatrix}.$$

Consider the vectors $\mathbf{u}_1 := \mathbf{Bv}_1, \mathbf{u}_2 = \mathbf{Bv}_2, \dots$. For each nonempty finite set $F \subseteq \mathbb{N}$, we have that

$$\mathbf{u}_F = \sum_{n \in F} \mathbf{u}_n = \sum_{n \in F} \mathbf{Bv}_n = \mathbf{B} \sum_{n \in F} \mathbf{v}_n = \mathbf{Bv}_F.$$

By the Central Sets Theorem, there are nonempty finite sets of natural numbers $F_1 < F_2 < \dots$ and elements $ab_1, ab_2, \dots \in C \cap a\mathbb{N}$ such that

$$\{\, ab_H + \mathbf{u}_{F_H} : H \in [\mathbb{N}]^{<\infty} \,\} = \mathrm{FS}(ab_1 + \mathbf{u}_{F_1}, ab_2 + \mathbf{u}_{F_2}, \dots) \subseteq (C \cap a\mathbb{N})^m \subseteq C^m.$$

Let

$$\mathbf{w}_1 := \begin{pmatrix} b_1 \\ \mathbf{v}_{F_1} \end{pmatrix}, \mathbf{w}_2 := \begin{pmatrix} b_2 \\ \mathbf{v}_{F_2} \end{pmatrix}, \dots .$$

For each nonempty finite set $H \subseteq \mathbb{N}$,

$$\mathbf{w}_H = \begin{pmatrix} b_H \\ \mathbf{v}_{F_H} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \begin{pmatrix} \vec{a} & \mathbf{B} \\ \vec{0} & \mathbf{A} \end{pmatrix} \mathbf{w}_H &= \begin{pmatrix} \vec{a} & \mathbf{B} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_H \\ \mathbf{v}_{F_H} \end{pmatrix} = \begin{pmatrix} ab_H + \mathbf{B}\mathbf{v}_{F_H} \\ \mathbf{A}\mathbf{v}_{F_H} \end{pmatrix} = \\ &= \begin{pmatrix} ab_H + \mathbf{u}_{F_H} \\ \mathbf{A}\mathbf{v}_{F_H} \end{pmatrix} \in C^{2m}. \end{aligned}$$

Thus, the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots \in \mathbb{N}^{n+1}$ are as required in the inductive claim. □

There is an obstacle for generalizing the Monochromatic Image Theorem to matrices $\mathbf{A}$ with arbitrary integer entries: If all entries of the matrix $\mathbf{A}$ are negative and $\mathbf{v} \in \mathbb{N}^n$, then all entries of the image vector $\mathbf{Av}$ are negative. Since we are given a coloring of $\mathbb{N}$, we must request that all entries of $\mathbf{Av}$ are natural numbers. It turns out that this is the only obstacle.

Theorem 2.4. *Let $\mathbf{A}$ be a rational $m \times n$ matrix with the first entries property, such that all first entries of $\mathbf{A}$ are positive. For each finite coloring of $\mathbb{N}$, there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that all entries of the image vector $\mathbf{Av}$ have the same color.*

Proof. Multiply the matrix $\mathbf{A}$ by a natural number a so that all entries of the matrix $\tilde{\mathbf{A}} := a\mathbf{A}$ are integer, and all first entries are greater than 1. Let N be a natural number greater than all absolute values of elements of the matrix $\tilde{\mathbf{A}}$. Let

$$\mathbf{B} := \begin{pmatrix} 1 & N & N^2 & \cdots & N^{n-1} \\ & 1 & N & \ddots & \vdots \\ & & \ddots & \ddots & N^2 \\ & O & & 1 & N \\ & & & & 1 \end{pmatrix}.$$

Consider the matrix $\tilde{\mathbf{A}}\mathbf{B}$. All elements of this matrix are in $\mathbb{N} \cup \{0\}$: For all i, j, let a_i be the first entry in row i of the matrix $\tilde{\mathbf{A}}$. For appropriate d, we have that

$$(\tilde{\mathbf{A}}\mathbf{B})_{ij} = a_i N^d + * \cdot N^{d-1} + \cdots + * \cdot 1$$

(or 0). Since $a_i > 1$ and the absolute value of each entry of $\tilde{\mathbf{A}}$ is smaller than N, we have that

$$\left| * \cdot N^{d-1} + \cdots + * \cdot 1 \right| \le (N-1)(N^{d-1} + N^{d-2} + \cdots + 1) = N^d - 1 < a_i N^d,$$

and thus the entry $(\tilde{\mathbf{A}}\mathbf{B})_{ij}$ is a positive integer.

The matrix $\tilde{\mathbf{A}}\mathbf{B}$ has the first entries property: The product of each row of the matrix $\tilde{\mathbf{A}}$ with the matrix $\mathbf{B}$ is of the form

$$(\underbrace{0, \dots, 0}_{k}, a_i, *, \dots, *) \begin{pmatrix} 1 & N & N^2 & \cdots & N^{n-1} \\ & 1 & N & \ddots & \vdots \\ & & \ddots & \ddots & N^2 \\ & O & & 1 & N \\ & & & & 1 \end{pmatrix} = (\underbrace{0, \dots, 0}_{k}, a_i, *, \dots, *),$$

and thus the first entries of the matrix $\tilde{\mathbf{A}}\mathbf{B}$ are equal to the first entries of the matrix $\tilde{\mathbf{A}}$, which has the first entries property.

By the Monochromatic Image Theorem, for each finite coloring of $\mathbb{N}$ there is a vector $\mathbf{v}$ of natural numbers such that the entries of the vector

$$\tilde{\mathbf{A}}\mathbf{B}\mathbf{v} = a\mathbf{A}\mathbf{B}\mathbf{v} = \mathbf{A}(a\mathbf{B}\mathbf{v})$$

have the same color. Each entry of the vector $\mathbf{B}\mathbf{v}$, a sum of products of natural numbers, is a natural number. Since a is a natural number, all entries of the vector $\mathbf{u} := a\mathbf{B}\mathbf{v}$ are natural. We have seen that the entries of the vector $\mathbf{A}\mathbf{u}$ are of the same color. □

THEOREM 2.5. *Let $\mathbf{A}$ be a rational $m \times n$ matrix with the first entries property, such that all first entries of $\mathbf{A}$ are positive. Assume, further, that the rows of $\mathbf{A}$ are distinct. For each finite coloring of $\mathbb{N}$, there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that the entries of the image vector $\mathbf{A}\mathbf{v}$ are distinct, and have the same color.*

PROOF. We may assume that every row i of $\mathbf{A}$ has a first entry a_i. For distinct rows $\mathbf{r}_i$ and $\mathbf{r}_j$ of $\mathbf{A}$, we have that $\mathbf{r}_i - \mathbf{r}_j \neq \mathbf{0}$. Assume that the first entry of the latter vector is in position k. Multiply this vector by a rational number q_{ij} such that its first entry becomes a_k, and add this new vector to the matrix $\mathbf{A}$ as a new row. We obtain a new rational matrix $\tilde{\mathbf{A}}$ with the first entries property, with all first entries positive.

By Theorem 2.4, there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that the entries of the vector $\tilde{\mathbf{A}}\mathbf{v}$ are natural and monochromatic. In particular, the entries of $\mathbf{A}\mathbf{v}$ are monochromatic, and for distinct rows $\mathbf{r}_i$ and $\mathbf{r}_j$ of $\mathbf{A}$, we have that $q_{ij}(\mathbf{r}_i - \mathbf{r}_j)\mathbf{v} \in \mathbb{N}$. Thus, $\mathbf{r}_i\mathbf{v} \neq \mathbf{r}_j\mathbf{v}$ for all i, j, that is, the entries of $\mathbf{A}\mathbf{v}$ are distinct. □

The following result follows immediately from the Monochromatic Image Theorem.

COROLLARY 2.6. *For all natural numbers n, c and k, for each finite coloring of $\mathbb{N}$, there are natural numbers $x_1, \dots, x_n$ such that all elements of all of the following sets are of the same color (where $[-k, k] := \{-k, -k+1, \dots, k-1, k\}$):*

$$\begin{array}{r} cx_1 + [-k,k]x_2 + [-k,k]x_3 + [-k,k]x_4 + \cdots + [-k,k]x_n \\ cx_2 + [-k,k]x_3 + [-k,k]x_4 + \cdots + [-k,k]x_n \\ cx_3 + [-k,k]x_4 + \cdots + [-k,k]x_n \\ \ddots \qquad \vdots \\ cx_n \end{array} .$$

(For example, in the first set there are $(2k+1)^{n-1}$ elements.) □

A straightforward modification of the proof of the Monochromatic Image Theorem gives the following.

THEOREM 2.7. *Let V be an infinite vector space over a field $\mathbb{F}$. Let $\mathbf{A}$ be an $m \times n$ matrix over $\mathbb{F}$ with the first entries property. For each finite coloring of $V \setminus \{\vec{0}\}$, there are vectors $v_1, \dots, v_n \in V \setminus \{\vec{0}\}$ such that all vectors*

$$a_{i1}v_1 + \cdots + a_{in}v_n,$$

for $i = 1, \dots m$, have the same color. □

EXERCISE 2.8. Prove Theorem 2.7.

3. Rado's Theorem for systems of linear equations

DEFINITION 3.1. A rational matrix $\mathbf{A} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ has the *columns property* if there is a partition $\{1, \dots, n\} = F_1 \cup \dots \cup F_k$, $k \geq 1$, such that:

(1) $\mathbf{v}_{F_1} = \mathbf{0}$.
(2) For each $i = 2, \dots, k$, the vector $\mathbf{v}_{F_i}$ is a linear combination over $\mathbb{Q}$ of the vectors $\mathbf{v}_j$, for $j \in F_1 \cup \dots \cup F_{i-1}$.

Following is a generalization of Rado's Theorem 7.3.4 to systems of linear equations.

THEOREM 3.2 (Rado). *Let $\mathbf{A}$ be a rational matrix with the columns property. For each finite coloring of $\mathbb{N}$, the homogeneous system of linear equations*

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

has a monochromatic solution.

PROOF. The assertion is not affected by changing the order of the columns of the matrix $\mathbf{A} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$, since one may also change the order of the entries of the solution accordingly. Thus, we may assume that

$$F_1 = \{1, \dots, n_1\}, F_2 = \{n_1 + 1, \dots, n_2\}, \dots, F_k = \{n_{k-1} + 1, \dots, n\}.$$

By the columns property, there are rational coefficients that, when placed instead of the asterisks below, make the equations hold:

$$\begin{array}{rcl} \mathbf{v}_1 + \dots + \mathbf{v}_{n_1} & = & \mathbf{0} \\ \mathbf{v}_{n_1+1} + \dots + \mathbf{v}_{n_2} & = & * \cdot \mathbf{v}_1 + \dots + * \cdot \mathbf{v}_{n_1} \\ & \vdots & \\ \mathbf{v}_{n_{k-1}+1} + \dots + \mathbf{v}_n & = & * \cdot \mathbf{v}_1 + \dots + * \cdot \mathbf{v}_{n_{k-1}}. \end{array}$$

Moving everything to the left hand side of the equations, this means that there are rational coefficients such that the following equations hold:

$$\underbrace{(\mathbf{v}_1, \dots, \mathbf{v}_n)}_{\mathbf{A}} \underbrace{\begin{pmatrix} 1 & * & & * \\ \vdots & \vdots & & \\ 1 & * & & \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \\ 0 & 1 & \cdots & \\ & 0 & & * \\ \vdots & \vdots & & 1 \\ & & & \vdots \\ 0 & 0 & & 1 \end{pmatrix}}_{\mathbf{B}} = O$$

The first entries of the right hand matrix $\mathbf{B}$ are all 1.

By the Monochromatic Image Theorem, for each finite coloring of $\mathbb{N}$ there is a vector $\mathbf{v}$ of natural numbers such that the entries of the vector $\mathbf{u} := \mathbf{B}\mathbf{v}$ are monochromatic. As

$$\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{B}\mathbf{v} = \mathbf{O}\mathbf{v} = \mathbf{0},$$

the vector $\mathbf{u}$ is a monochromatic solution of the system $\mathbf{A}\mathbf{x} = \mathbf{0}$. $\square$

The proof shows that, for each central set, all system $\mathbf{A}\mathbf{x} = \mathbf{0}$ for rational matrices $\mathbf{A}$ with the columns property have solutions with all entries in the central set. It follows that, for each finite coloring of $\mathbb{N}$, all such systems have monochromatic solutions of the same color.

4. Comments for Chapter 8

The formulation and proof of the Piecewise Syndetic Sets Theorem (Theorem 7.2.3) should be considered a part of the proof of the Central Sets Theorem (Theorem 1.3). The Central Sets Theorem was first proved, using a different but equivalent notion of "central set" (Theorem 19.27 in Hindman–Strauss), in Hillel Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981. The method used in the present proof of this theorem is from Hillel Furstenberg and Yitzhak Katznelson, *Idempotents in compact semigroups and Ramsey Theory*, Israel Journal of Mathematics, 1989. Their proof was converted to the one included here by Vitaly Bergelson and Neil Hindman (*Nonmetrizable topological dynamics and Ramsey Theory*, Transactions of the American Mathematical Society, 1990).

Corollary 2.6 is due to Walter Deuber, *Partitionen and lineare Gleichungssysteme*, Mathematische Zeitschrift, 1973. Theorem 2.7 is due to Vitaly Bergelson, Walter Deuber and Neil Hindman, *Rado's Theorem for finite fields*, Colloquia Mathematica Societatis János Bolyai, 1992.

The converse of Theorem 3.2 also holds. The proof is similar in nature to the one presented here in the case of a single linear equation. The interested reader is referred to the Hindman–Strauss monograph for details.

The following theorem is proved in Neil Hindman and Imre Leader, *Image partition regularity of matrices*, Combinatorics, Probability and Computing, 1993. The condition in this theorem is more general than the first entries property.

THEOREM 4.1. *Let $\mathbf{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an $m \times n$ rational matrix. The following assertions are equivalent:*

1. *The matrix $\mathbf{A}$ has monochromatic images for every finite coloring of $\mathbb{N}$.*
2. *There are positive rational numbers $a_1, \ldots, a_n$ such that the matrix*
$$(a_1\mathbf{v}_1, \ldots, a_n\mathbf{v}_n, -\mathbf{e}_1, \ldots, -\mathbf{e}_m)$$
has the columns property.

EXERCISE 4.2. Determine which of the following matrices has monochromatic images for every finite coloring of $\mathbb{N}$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & -9 \\ 2 & -2 & 3 \end{pmatrix};\ \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}.$$

In order to genearlize the full-fledged Hindman Theorem in this framework, one may consider *infinite* matrices with finitely supported rows, that is, such that all but finitely many entries in each row are zero. While much has been proved in this general setting, we are still far from a complete characterization as in Theorem 4.1. The Hindman–Strauss monograph contains an account of this question, that is still actively investigated.

CHAPTER 9

Qualitative versions of Ramsey's Theroem

1. Superfilters

Most theorems of the earlier chapters assert that, for each finite coloring of $\mathbb{N}$, there is a monochromatic structure of a certain type: an FS set, long arithmetic progressions, image vectors of matrices, or solutions of systems of equations. In many cases, we have seen that even in finite colorings of sets smaller than $\mathbb{N}$ one may find the same type of monochromatic structures. For example, for each finite coloring of an FS set there is a monochromatic FS subset. The reason is that FS sets are characterized as members of idempotent ultrafilters. For the same reason, the analogous assertion for piecewise syndetic sets holds. The finite version of van der Waerden's Theorem (Exercise 2.9) implies the same assertion for sets containing arbitrarily long arithmetic progressions.

Exercise 1.1. Let $A \subseteq \mathbb{N}$ be a set containing arbitrarily long arithmetic progressions. Prove that, for each finite coloring of A, there is a monochromatic subset of A containing arbitrarily long arithmetic progressions.

This phenomenon motivates the following definition.

Definition 1.2. A nonempty family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is a *superfilter* if, for each set $A \in \mathcal{A}$ and each finite coloring of A, there is a monochromatic set $B \subseteq A$ in $\mathcal{A}$.

Thus, for example, the families of all infinite subsets of $\mathbb{N}$, all FS sets, all sets containing arbitrarily long arithmetic progressions, and all syndetic sets, are superfilters.

Notice that if $\mathcal{S}$ is a superfilter and $A \in \mathcal{S}$, then every set $B \supseteq A$ is also in $\mathcal{S}$. Also, if a union of two sets, $A \cup B$, is in $\mathcal{S}$ then A or B are in $\mathcal{S}$.

Exercise 1.3. For a family $\mathcal{B} \subseteq [\mathbb{N}]^\infty$, let $\operatorname{Mono}(\mathcal{B})$ be the family of all sets $A \subseteq \mathbb{N}$ such that, for each finite coloring of A, there is a monochromatic set $B \subseteq A$ in $\mathcal{B}$. Prove the following assertions:

(1) The family $\operatorname{Mono}(\mathcal{B})$ is nonempty if and only if $\mathbb{N} \in \operatorname{Mono}(\mathcal{B})$.
(2) If $\mathbb{N} \in \operatorname{Mono}(\mathcal{B})$, then the family $\operatorname{Mono}(\mathcal{B})$ is a superfilter.

Every nonprincipal ultrafilter is a superfilter, and so is any union of ultrafilters. This is, in fact, a characterization of superfilters.

Lemma 1.4. *Let $\mathcal{S}$ be a superfilter, and let $C = \{\, p_\alpha : \alpha \in I \,\}$ be the set of all ultrafilters p with $p \subseteq \mathcal{S}$. Then C is a nonempty closed subset of $\beta\mathbb{N} \setminus \mathbb{N}$, and $\mathcal{S} = \bigcup_{\alpha \in I} p_\alpha$.*

Proof. Let $\mathcal{F} = \{\, A \subseteq \mathbb{N} : A^{\mathsf{c}} \notin \mathcal{S} \,\}$. Then $\mathcal{F}$ is a filter: The set $\mathbb{N}$ is in $\mathcal{F}$ since its complement, $\emptyset$, is finite. If $A, B \in \mathcal{F}$ and $A \cap B \notin \mathcal{F}$, then $(A \cap B)^{\mathsf{c}} \in \mathcal{S}$, that is, $A^{\mathsf{c}} \cup B^{\mathsf{c}} \in \mathcal{S}$. It follows that A^{c} or B^{c} are in $\mathcal{S}$, in contradiction to $A, B \in \mathcal{F}$.

For each finite set $F \subseteq \mathbb{N}$, since $F \cup F^{\mathsf{c}} \in \mathcal{S}$ and $F \notin \mathcal{S}$, we have that $F^{\mathsf{c}} \in \mathcal{S}$. Thus, there are no finite sets in $\mathcal{F}$.

Let $A \in \mathcal{S}$. For each $B \in \mathcal{F}$, the set $B \cap A$ is in $\mathcal{S}$: Since $A \subseteq B^{\mathsf{c}} \cup (A \cap B)$, the latter set is in $\mathcal{S}$. Since $B^{\mathsf{c}} \notin \mathcal{S}$, we have that $A \cap B \in \mathcal{S}$. In particular, the set $B \cap A$ is infinite.

As $\mathcal{F}$ is a filter, the family $\{ B \cap A : B \in \mathcal{F} \}$ is closed under finite intersections. Thus, this family extends to a nonprincipal ultrafilter p on $\mathbb{N}$. Then $A \in p$, and it remains to show that $p \subseteq \mathcal{S}$. Let $B \in p$. If $B \notin \mathcal{S}$, then $B^{\mathsf{c}} \in \mathcal{F}$, and then $B^{\mathsf{c}} \cap A \in p$. It follows that $B^{\mathsf{c}} \in p$; a contradiction. $\square$

2. Eventually monochromatic sets

All theorems below can be proved for colorings of $[\mathbb{N}]^d$, where d is an arbitrary natrural number. We present them for the most interesting, two-dimensional case. The proofs in this case contain all ingredients needed for inducting on d.

Definition 2.1. Let c be a finite coloring of the complete graph $[\mathbb{N}]^2$. A complte subgraph $[A]^2$ of $[\mathbb{N}]^2$ is *eventually monochromatic* if there is a color i such that, for each $a \in A$, the color of the edges $\{a, b\}$ is i for all but finitely many $b \in A$. For families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{N}$, the notation

$$\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2$$

denotes the statement that, for each $A \in \mathcal{A}$ and each finite coloring of A, there is a set $B \subseteq A$ in $\mathcal{B}$ that is eventually monochromatic.

The partition relation in Definition 2.1 is due to James E. Baumgartner and Alan D. Taylor, *Partition Theorems and Ultrafilters*, Transactions of the American Mathematical Society, 1978.

The assertion $\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2$ becomes stronger if the family $\mathcal{A}$ is enlarged or the family $\mathcal{B}$ is thinned out.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of $\mathbb{N}$. If $\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2$ then, for each set $A \in \mathcal{A}$ and each finite coloring of A, there is a monochromatic set $B \subseteq A$ in $\mathcal{B}$. In particular, if $\mathcal{A} \longrightarrow \lfloor \mathcal{A} \rfloor^2$, then the family $\mathcal{A}$ is a superfilter.*

Proof. Let $A \in \mathcal{A}$, and let c be a finite coloring of A. Define a coloring χ of $[A]^2$ by

$$\chi(\{a, b\}) := c(\min\{a, b\}).$$

Assume that an infinite set $B \subseteq A$ is eventually monochromatic, say of color i. For each $b \in B$, taking a large enough element $b' \in B$, we have that $c(b) = \chi(\{b, b'\}) = i$. $\square$

Exercise 2.3. Show that a 2-coloring suffices in the proof of Proposition 2.2.
Hint: Color a pair $\{a, b\}$ according to whether $c(a) = c(b)$ or not.

Exercise 2.4. For a natural number $k \geq 2$, write $\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2_k$ if, for each $A \in \mathcal{A}$ and each k-coloring of A, there is a set $B \subseteq A$ in $\mathcal{B}$ that is eventually monochromatic. Consider the case where $\mathcal{B} = \mathcal{A}$. Prove that the statements $\mathcal{A} \longrightarrow \lfloor \mathcal{A} \rfloor^2_k$, for $k \geq 2$, are all equivalent.
Hint: A color blindness argument.

We will soon establish a simple sufficient condition for the relation $\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2$ to hold. The proof will use the following lemma.

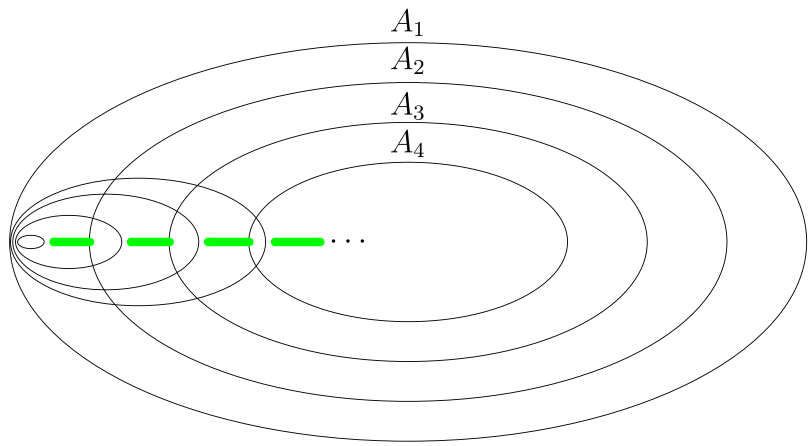
Lemma 2.5. *Let p be a nonprincipal ultrafilter on $\mathbb{N}$. For each finite coloring of the complete graph $[\mathbb{N}]^2$, there are sets $A_1 \supseteq A_2 \supseteq \cdots$ in p such that all edges $\{i, j\}$ with $i \in A_1$ and $j \in A_i$ have the same color.*

Proof. The proof is similar to the ultrafilter proof of Ramsey's Theorem. For an element $v \in \mathbb{N}$ and a color i, let $C_i(v)$ be the set of all vertices connected to v by an edge of color i. Then $A \setminus \{v\} = C_1(v) \cup \cdots \cup C_k(v) \in p$, and there is i with $C_i(v) \in p$. Defining $\chi(v) := i$, we obtain a finite coloring χ of A. Fix a set $A_1 \in p$ that is monochromatic for the coloring χ, say of the color green. We henceforth write $C(v)$ for $C_i(v)$. Thus, for each $v \in A_1$, we have that $C(v) \in p$ and all edges $\{v, u\}$, for $u \in C(v)$, are green.

For each $n > 1$, let

$$A_n = \begin{cases} A_{n-1} \cap C(n) & n \in A_1 \\ A_{n-1} & \text{otherwise.} \end{cases}$$

This construction is illustrated by the following figure. In this figure, the left hand ellipses denote the sets $A_1 \cap \{1\}$, $A_1 \cap \{1, 2\}$, $A_1 \cap \{1, 2, 3\}$, etc, that add up to cover all elements of A_1.



It is clear from the diagram that the requested assertion follows. $\square$

Lemma 2.5 is proved in Vitaly Bergelson and Neil Hindman, *Ultrafilters and multidimensional Ramsey theorems*, Combinatorica, 1989.

DEFINITION 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be a families of subsets of $\mathbb{N}$. $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ is the assertion that, for each sequence $A_1, A_2, \dots \in \mathcal{A}$, we can select finite sets $F_1 \subseteq A_1, F_2 \subseteq A_2, \dots$ such that the set $\bigcup_n F_n$ is in $\mathcal{B}$.

For example, let $[\mathbb{N}]^\infty$ be the family of all infinite subsets of $\mathbb{N}$. Then $\mathsf{S}_{\text{fin}}([\mathbb{N}]^\infty, [\mathbb{N}]^\infty)$ holds. More interestingly, let AP be the family of all subsets of $\mathbb{N}$ with arbitrarily long arithmetic progressions. Say that a set $A \subseteq \mathbb{N}$ is *Rado* if it contains a solution for all rational homogeneous systems of linear equations with the columns property. Let $\mathcal{C}$ be the family of central sets, and $\mathcal{R}$ be the family of Rado sets. Then $\mathsf{S}_{\text{fin}}(\mathcal{C}, \mathcal{R})$ holds. There are examples where $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{A})$ holds but not for a trivial reason. These will be mentioned later.

THEOREM 2.7. *Let $\mathcal{A}$ be a superfilter and $\mathcal{B}$ be a family of subsets of $\mathbb{N}$ such that $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ holds. Then $\mathcal{A} \longrightarrow \lfloor \mathcal{B} \rfloor^2$.*

PROOF. Let $A \in \mathcal{A}$. By Lemma 1.4, there is a nonprincipal ultrafilter p on $\mathbb{N}$ with $A \in p \subseteq \mathcal{A}$. Let $A_1 \supseteq A_2 \supseteq \cdots$ be as in Lemma 2.5. Intersecting each of these sets with A, we may assume that $A_1 \subseteq A$. By $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$, there are finite sets $F_1 \subseteq A_1, F_2 \subseteq A_2, \dots$ such that the set $B := \bigcup_n F_n$ is in $\mathcal{B}$. Then the set B is eventually monochromatic. $\square$

DEFINITION 2.8. Let c be a finite coloring of the graph $[\mathbb{N}]^2$. A complete subgraph $[A]^2$ of $[\mathbb{N}]^2$ is *monochromatic modulo finite* if there is a partition of A into finite sets such that all edges among distinct pieces of the partition have the same color. For families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{N}$, the notation

$$\mathcal{A} \longrightarrow [\mathcal{B}]^2$$

denotes the statement that, for each $A \in \mathcal{A}$ and each finite coloring of A, there is a set $B \subseteq A$ in $\mathcal{B}$ that is monochromatic modulo finite.

If B is monochromatic modulo finite for a coloring of $\mathbb{N}$, then it is in particular eventually monochromatic. The converse implication need not hold. To see that, color an edge $\{i, j\} \in [\mathbb{N}]^2$ red if $j \leq 2i$ and green otherwise. Then the graph is eventually green, but there is a red path between every pair of vertices.

For natural numbers $m < n$, let $[m, n) = \{m, m+1, \cdots, n-1\}$. The following lemma is more than needed for the subsequent theorem, but it will be used again later.

Lemma 2.9. *Let $\mathcal{B}$ be a family of infinite subsets of $\mathbb{N}$ with the property that, for each $B \in \mathcal{B}$ and each increasing sequence $m_1 = 1 < m_2 < \cdots$, there is a subsequence $l_1 = 1 < l_2 < \cdots$ such that $\bigcup_n B \cap [l_{2n-1}, l_{2n}) \in \mathcal{B}$ or $\bigcup_n B \cap [l_{2n}, l_{2n+1}) \in \mathcal{B}$. Assume that, for a given coloring of $[\mathbb{N}]^2$, the complete subgraph $[B]^2$ is eventually monochromatic. Then there is a subset $C \subseteq B$ in $\mathcal{B}$ such that the complete graph $[C]^2$ is monochromatic modulo finite.*

Proof. Assume that the eventual color of the graph $[B]^2$ is green. Define an increasing function $f\colon \mathbb{N} \to \mathbb{N}$ by induction on n, such that $f(n) > n$ for all n and, for each $n \in B$, the edge $\{n, m\}$ is green for all $m \geq f(n)$.

Set $m_1 = 1$, and for each $n > 1$, let $m_n = f(m_{n-1})$. By thinning out the sequence, we may assume that all sets $B \cap [m_n, m_{n+1})$ are nonempty. Let $l_1 = 1 < l_2 < \cdots$ be a subsequence as in the premise of the lemma. Since this is a subsequence of $m_1, m_2, \ldots$, it remains the case that $f(l_n) \leq l_{n+1}$ for all n.

Assume, for example, that $\bigcup_n B \cap [l_{2n}, l_{2n+1}) \in \mathcal{B}$. Then all edges among distinct sets $B \cap [l_{2n}, l_{2n+1})$ are green. Indeed, let $a \in B \cap [l_{2n}, l_{2n+1})$ and $b \in B \cap [l_{2m}, l_{2m+1})$, for $n < m$. Then $f(a) \leq f(l_{2n+1}) \leq l_{2n+2} \leq l_{2m}$. Thus, the edge $\{a, b\}$ is green. $\square$

Theorem 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be superfilters such that $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ holds. Then $\mathcal{A} \longrightarrow [\mathcal{B}]^2$.*

Proof. Given a finite coloring of a set $A \in \mathcal{A}$, there is by Theorem 2.7 a subset $B \subseteq A$ in $\mathcal{B}$ such that the subgraph $[B]^2$ is eventually monochromatic. Since $\mathcal{B}$ is a superfilter, the assumption of Lemma 2.9 is satisfied for all subsequences. It follows that there is a set $C \subseteq B$ in $\mathcal{B}$ such that the graph $[B]^2$ is monochromatic modulo finite. $\square$

Corollary 2.11. *Let* AP *be the family of all subsets of $\mathbb{N}$ containing arbitrarily long arithmetic progressions. Then* $\mathrm{AP} \longrightarrow [\mathrm{AP}]^2$.

Proof. The family AP is a superfilter, and $\mathsf{S}_{\text{fin}}(\mathrm{AP}, \mathrm{AP})$ holds. Apply Theorem 2.10. $\square$

Corollary 2.11 can be reformulated as follows.

Corollary 2.12. *Let $A \subseteq \mathbb{N}$ be a set containing arbitrarily long arithmetic progressions. For each finite coloring of $[A]^2$, there are for each n a monochromatic arithmetic of length n, such that the edges among elements of distinct arithmetic progressions are all of the same color.*

Proof. Given a set $B \subseteq A$ that is monochromatic modulo finite and a partition $B = \bigcup_n F_n$ witnessing that, we can move to a coarser partition by choosing, for each n, a number m_n so large such that the set $H_n := F_{m_{n-1}} \cup \cdots F_{m_n}$ contains an arithmetic progression of length n. Edges among distinct sets H_n are of the same color, being edges among distinct pieces of the original partition. $\square$

The case $\mathcal{A} = \mathcal{B}$ of Theorem 2.10, is proved in Nadav Samet and Boaz Tsaban, *Superfilters, Ramsey theory, and van der Waerden's Theorem*, Topology and its Applications, 2009. Corollary 2.11 was first proved in the same paper, by a direct method. The proof given here is slightly simpler. A weaker form of this corollary, where only finite colorings of the entire set $\mathbb{N}$ are considered, is proved in Vitaly Bergelson and Neil Hindman, *Ultrafilters and multidimensional Ramsey theorems*, Combinatorica, 1989.

3. Trying

Since $U \cup \mathcal{V} := \{ U \cup V : V \in \mathcal{V} \} \in \Omega$ for $\mathcal{V} \in \Omega$, the ultras contained in $\{ \mathcal{V} \subseteq \mathcal{U} : \mathcal{V} \in \Omega \}$ is a left ideal in $\beta(FU(\mathcal{U}), \cup)$. Let e be a minimal idempotent there.

Let green be the e-prefered color of the graph edges, and $A \in e$ with all preferring green.

e ip, so there is $A' \subseteq A$ in e that "can be continued": Each $a \in A'$ has $B \in e$ with $a+B \subseteq A'$.

Game proof:

(1) Fix $A_1 \in e$ of points preferring green.
(2) Select $a_1 \in A_1^\star := \{ a \in A_1 : \exists B \in e, a + B \subseteq A_1 \}$.
(3) $A_2 \subseteq A_1$ green neighbors of a_1 with $a_1 + A_2 \subseteq A_1$.
(4) Select $a_2 \in A_2^\star$.
(5) $A_3 \subseteq A_2$ in e, green neighbors of all $\mathrm{FS}(a_1, a_2)$. (Possible since $\mathrm{FS}(a_1, a_2) \subseteq A_1$!). Request also $E(a_2, A_3)$ green.
(6) etc.

$\{a_1, \dots\} \in \Omega$.

By $\mathsf{S}_{\mathrm{fin}}$, there is $B \subseteq^* A_n$ in Ω. Color $\{b, c\}$ green if $b + c \in A$. Since eventually green and $\mathsf{S}_1(\Omega, \Omega)$ gives partition relation, there is a subgraph in Ω that is all green.

Will consider the finite sums of

***

4. Todo

S_1 implies game using idea of abstract uf ramsey and partitioning. Requires just the extra step of Q-point/partitioning to intervals. DOES NOT NEED THAT $\mathcal{A}$ is a superfilter. Also the interval partition on $\mathcal{B}$ needs only the technical property.

Exemplify above with $\mathcal{F}_n$'s etc., as below.

Add addition and get ip based theorems as below.

(If Ω won't be ideal, then consider other ideals.) Can get MT-like from $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$: Qualitative MT: 1. Using games: e idempotent (e.g. for Ω), e prefers green edges, $A \in e$ preferring green. B_1 - those having sum in A with a large $A_1 \subseteq A$. Game: Take $F_1 \subseteq A$, A_1 good for all F_1. Continue. Get ω-cover. Apply Q-point.

Game free: Let $A \in e$. There is $A' \subseteq A$ in e s.t. for each $a \in A'$ there is $B \subseteq A'$ in e with $a + B \subseteq A$. Given A, move to A' and choose for each $a \in A'$ the B' instead of B. Then do as in ufRamsey proof, for A', A'_1,.... Apply $\mathsf{S}_{\mathrm{fin}}$. All elements of A' were continued. Apply Q-pt to $B \cap (A_{n+1} \setminus A_n)$. All FS are in A' so were considered and can do interval partitioning so that next m_n is good for $FS(B \cap [1, m_{n-1}))$.

...

$\mathsf{S}_{\mathrm{fin}}(e, \Omega)$ to get eventually mono graph in Ω.

Paritioning trick, step n: Look at $B \cap [1, m_n)$.

for increasing sets of finite sums of finite sets s.t. remain in Ω and edges among $FS(B \cap [m_5, m_6))$ and $FS(A \cap [m_7, m_8))$ green.

5. Finite sums of arithmetic progressions, images, and solutions

Following is an abstract version of the Central Sets Theorem.

The following can be generalized to colorings of $[\mathbb{N}]^d$. This is stronger as d increases, since can color by minimal element. (Think what happens for $d > 2$.) And this would generalize also Milliken–Taylor.

Essentially this is the Combinatorica result.

Better leave as now, then hybrid-of-hybrids (since doesn't extend the second abs2, which doesn't need an ip). Or do hybhyb now but add the abs2 anyway.

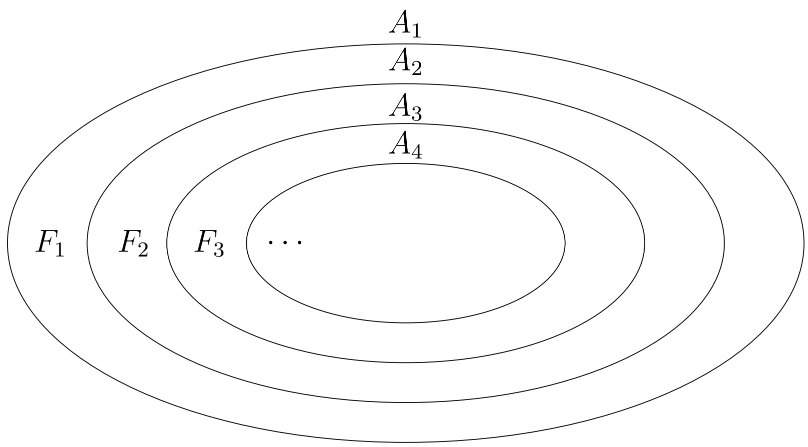
To do: Is there infinite-dim version of the abs thm? (As there is for ramsey).

THEOREM 5.1. *Let $(S,+)$ be a semigroup. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be families of nonempty finite subsets of S. Assume that there is an idempotent $e \in \beta S$ such that every element of e has a subset in each $\mathcal{F}_n$. Then, for each $A \in e$, there are sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \dots$ such that every finite sum of elements of the set $\bigcup_n F_n$, with at most one element from each F_n, is in A. In other words, such that*

$$F_{i_1} + \cdots + F_{i_k} \subseteq A$$

for all k and all $i_1 < \cdots < i_k$.

PROOF. The proof is the essence of the proof of the Central Sets Theorem.



Let $A_1 := A$.

For $n = 1, 2, \dots$, do the following: Take from the family $\mathcal{F}_n$ a subset F_n of A_n. Since e is an idempotent and $A_n \in e$, for each $a \in F_n$ there is in e a set B_a such that $a + B_a \subseteq A_n$. Let $A_{n+1} := \bigcap_{a\in F} B_a$. Then $A_n \in e$, and $F_n + A_{n+1} \subseteq A_n$.

The proof that all requested finite sums are in A is as in the proof of Hindman's Theorem. $\square$

COROLLARY 5.2. *Let $(S,+)$ be a semigroup. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be families of nonempty finite subsets of S. Assume that there is an idempotent $e \in \beta S$ such that, for each set $A \in e$ and each n, there is in $\mathcal{F}_n$ a subset of A. For each finite coloring of $\mathbb{N}$ there are sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \dots$ such that all finite sums of elements of the set $\bigcup_n F_n$, with at most one element from each F_n, have the same color.* $\square$

Consider the trivial case where, for each n, $\mathcal{F}_n$ is the family of all singletons (one element subsets of $\mathbb{N}$). In this case, the premise of the theorem holds for any idempotent ultrafilter e, and we obtain Hindman's Theorem.

For a vector

$$\mathbf{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in (S \cup \{0\})^m,$$

let

$$\mathrm{Set}(\mathbf{v}) := \{a_1, \dots, a_m\},$$

the set of entries of the vector $\mathbf{v}$. Let $(S,+)$ be an abelian semigroup, $\mathbf{v}_1, \mathbf{v}_2, \dots \in (S \cup \{0\})^m$, and $C \subseteq S$ be a central set. For each n, let

$$\mathcal{F}_n = \{\, \mathrm{Set}(a + \mathbf{v}_F) : a \in A, F > n \,\}.$$

Let $e \in \beta S$ be a minimal idempotent with $C \in e$. By the Piecewise Syndetic Sets Theorem, for each $A \in e$ there is in $\mathcal{F}_n$ a subset of A. By Theorem 5.1, there are sets

$$\mathrm{Set}(a_1 + \mathbf{v}_{F_1}) \in \mathcal{F}_1, \mathrm{Set}(a_2 + \mathbf{v}_{F_2}) \in \mathcal{F}_2, \dots$$

such that every finite sum of elements of the set $\bigcup_n \mathrm{Set}(a_n + \mathbf{v}_{F_n})$, with at most one element from each set $\mathrm{Set}(a_n + \mathbf{v}_{F_n})$, is in C. It follows that

$$\mathrm{FS}(a_1 + \mathbf{v}_{F_1}, a_2 + \mathbf{v}_{F_2}, \dots) \subseteq C^m.$$

By moving to a subsequence of the given sets, we may assume that $F_1 < F_2 < \cdots$. We obtain the Central Sets Theorem.

The following theorem is a simultaneous generalization of van der Waerden's Theorem and Hindman's Theorem.

THEOREM 5.3. *For each finite coloring of $\mathbb{N}$, there are for each n an arithmetic progression of length n such that all finite sums of elements of these progressions, with at most one element from each progression, have the same color.*

PROOF. For each n, let $\mathcal{F}_n$ be the family of all arithmetic progressions of length n. Fix a minimal idempotent $e \in \beta\mathbb{N}$. For each $A \in e$, the set A is piecewise syndetic and thus has a subset in every $\mathcal{F}_n$. The assertion then follows, by Theorem 5.1. □

Following is a more general form of Theorem 5.3. To state it succinctly, say that a subset I of $\mathbb{N}$ is an *image set* of an $m \times n$ matrix $\mathbf{A}$ with the first entries property if there is a vector $\mathbf{v} \in \mathbb{N}^n$ such that I is the set of entries of the vector $\mathbf{A}\mathbf{v}$. An representative example of an image set is provided in Theorem 2.6.

THEOREM 5.4. *For each finite coloring of $\mathbb{N}$, there are for each matrix $\mathbf{A}$ with the first entries property an image set such that all finite sums of elements of these image sets, choosing at most one element from each image set, have the same color.*

PROOF. Let $\mathbf{A}_1, \mathbf{A}_2, \dots$ enumerate all matrices with the first entries property. For each n, let $\mathcal{F}_n$ be the family of all image sets of the matrix $\mathbf{A}_n$. By the Monochromatic Image Theorem, Theorem 5.1 applies. □

A simultaneous generalization of Schur's Coloring Theorem and Hindman's Theorem is, simply, Hindman's Theorem. But we also have a simultaneous generalization of Rado's Theorem and Hindman's Theorem.

THEOREM 5.5. *For each finite coloring of $\mathbb{N}$, there are for each system of linear equations over $\mathbb{N}$ with the columns property a solution such that all finite sums of entries of these solutions, choosing at most one entry from each solution, have the same color.*

PROOF. Theorems 8.3.2 and 5.1. □

We can combine any finite number of theorems whose proofs via Theorem 5.1 requires the same type of ultrafilter (in the above cases, a minimal idempotent). For example, let

$$(\mathbf{A}_1, \mathbf{B}_1), (\mathbf{A}_2, \mathbf{B}_2), \dots$$

enumerate all pairs of matrices such that the first matrix has the first entries property and the second matrix has the columns condition. For each n, put $F \in \mathcal{F}_n$ if and only if F contains an image set of the matrix $\mathbf{A}_n$ and the set of entries of a solution of the system $\mathbf{B}_n\mathbf{x} = \mathbf{0}$. Then Theorem 5.1 applies with any minimal idempotent e, and we obtain the following result: Let a finite coloring of $\mathbb{N}$ be given. There are, for each pair $(\mathbf{A}, \mathbf{B})$ of a matrix with the first entries condition and a matrix with the colums condition, a set containing an image of $\mathbf{A}$ and the entries of a solution of $\mathbf{Bx} = \mathbf{0}$ such that all finite sums choosing at most one element from each of these sets have the same color.

EXERCISE 5.6. Show that Theorem 5.1 holds for arbitrary families $\mathcal{F}_n$ of sets for which there is an idempotent $e \in \beta S$ such that, for each set $A \in e$ and each n, there is in $\mathcal{F}_n$ a subset of A that is not in e.

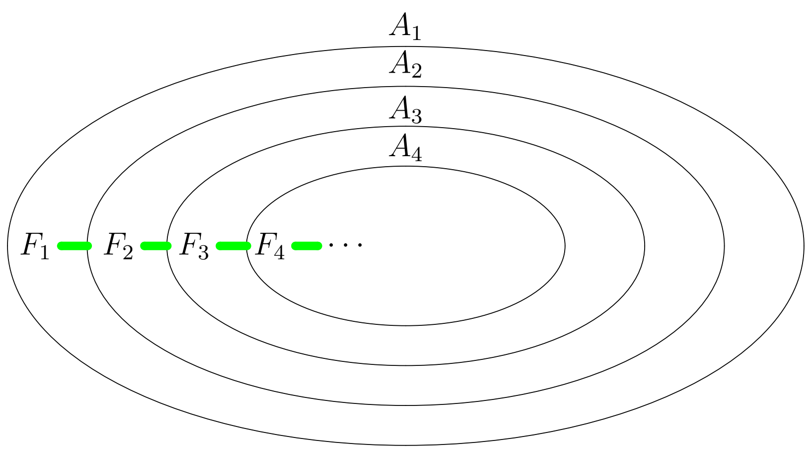
6. Ramsey's Theorem with anything else: Milliken–Taylor and beyond

In Section 2.2, we described an ultrafilter proof of Ramsey's Theorem. A nice feature of this proof was that it did not assume anything about the chosen ultrafilter. This makes it possible to use the method of the previous section in a broader setting, and obtain simultaneous generalizations of various theorems and Ramsey's Theorem. We begin with an abstract theorem.

THEOREM 6.1. *Let V be set and d be a natural number. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be families of nonempty finite subsets of V. Assume that there is a nonprincipal ultrafilter $p \in \beta V$ such that each element of p has a subset in each $\mathcal{F}_n$. Let $A \in p$. Then, for each finite coloring of an $[A]^d$, there are sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \dots$ such that all sets in $\left[\bigcup_n F_n\right]^d$ with at most one element from each F_n have the same color.*

PROOF. We treat the case $d = 2$, which is the most appealing visually. In this case, a finite coloring of the complete graph $[A]^2$ is given, and we find disjoint sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \dots$ such that all edges among distinct sets F_n have the same color. The generalization to larger d follows, in a similar manner, by induction on d.

For each $v \in A$ and each color i, let $A_i(v)$ be the set of all vertices connected to v by an edge of color i. Then $A \setminus \{v\} = A_1(v) \cup \cdots \cup A_k(v) \in p$, and thus there is i with $A_i(v) \in p$. Defining $\chi(v) := i$, we obtain a finite coloring χ of A. Fix a set $A_1 \in p$ that is monochromatic for the coloring χ, say of color i. We proceed as follows:



Take $F_1 \in \mathcal{F}_1$ such that $F_1 \subseteq A_1$. Let $A_2 := A_1 \cap \left(\bigcap_{v \in F_1} A_i(v)\right)$. Then $A_2 \in p$, and $F_1 \cap A_2 = \emptyset$.

Take $F_2 \in \mathcal{F}_2$ such that $F_2 \subseteq A_2$. Let $A_3 := A_2 \cap \left(\bigcap_{v \in F_2} A_i(v)\right)$. Then $A_3 \in p$, and $F_2 \cap A_3 = \emptyset$.

Continue in the same manner. $\square$

An *AP set* is a subset of $\mathbb{N}$ containing arbitrarily long arithmetic progressions.

THEOREM 6.2. *Let $A \subseteq \mathbb{N}$ be a set containing arbitrarily long arithmetic progressions. For each finite coloring of $[A]^d$, there are for each n an arithmetic progression of length n such that all d-element sets, consisting of elements of distinct progressions, have the same color.*

Theorem 6.2 follows from Theorem 6.1 and the following lemma.

Definition 6.3. A family of sets $\mathcal{F}$ is *partition-regular* if for each $A \in \mathcal{F}$ and every finite coloring of A, there is a monochromatic subset of A in $\mathcal{F}$.

There are AP sets which are not syndetic.

By the finite version of van der Waerden's Theorem, the family of all AP subsets of $\mathbb{N}$ is partition regular.

Lemma 6.4. *Let X be a set, and $\mathcal{F} \subseteq P(X)$ be a partition-regular family. For each $A \in \mathcal{F}$, there is an ultrafilter p on X such that $A \in p \subseteq \mathcal{F}$.*

Proof. *** $\square$

*** Elements of minimal elements of $\beta\mathbb{N}$ containt arithmentic progressions of all finite lengths. (In particular, minimal ultrafilters are nonprincipal.)

Exercise 6.5. Formulate theorems analogous to ones of the previous section, using Theorem 6.1.

The following theorem generalizes Theorem 5.1. Modulo its stronger hypothesis, this theorem also generalizes Theorem 6.1.

Theorem 6.6. *Let $(S,+)$ be a semigroup and d be a natural number. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be families of nonempty finite subsets of S. Assume that there is an idempotent $e \in \beta S$ such that each element of e has a subset in each $\mathcal{F}_n$. Let $A \in e$. Then, for each finite coloring of an $[A]^d$, there are sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \dots$ such that, for all finite nonempty index sets $I_1 < I_2 < \cdots < I_d$ and all elements*

$$a_1 \in \sum_{i \in I_1} F_i, a_2 \in \sum_{i \in I_2} F_i, \dots, a_d \in \sum_{i \in I_d} F_i,$$

the sets $\{a_1, \dots, a_d\}$ have the same color.

Proof. A straightforward combination of the proof of Theorem 5.1 and Theorem 6.1. $\square$

Theorem 6.7 (Milliken–Taylor). *Let $(S,+)$ be a semigroup and d be a natural number. Then, for each finite coloring of $[S]^d$, there are $a_1, a_2, \dots \in S$ such that, for all finite nonempty index sets $I_1 < I_2 < \cdots < I_d$, the sets $\{a_{I_1}, \dots, a_{I_d}\}$ have the same color.*

Proof. In Theorem 6.6, take all $\mathcal{F}_n$ to be the family of all singletons, and let e be an arbitrary idempotent of βS. $\square$

In HS, they talk about sum-subsystem in MT. They also mention their theorem is not general enouch to get the product version directly. Check whether our approach does.

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Application: Bergelson–Hindman Combinatorica.

We have as corollary the case of those "partition regulars defined by finite sets". Using they contain an uf. First time in ST. Maybe better since inner, no mention of uf.

Add the result when we only color an AP set. For this, we need that every partition-regular family, each set in the family is in an uf.

Milliken–Taylor (Hindman and Ramsey). Uses the game approach: (a_1, A_2),...,$(FS(a_1, .., a_n), A_{n+1})$, such that $FS(a_1, .., a_n) + A_{n+1} \subseteq A_n$ and $FS(a_1, .., a_n) \sim A_{n+1}$ is green.

Section on SPM.

Ramsey's Theorem begins this book and ends it. We have closed a circle.

7. Comments for Chapter 9

Theorem 5.4, which generalizes Theorem 5.3, is proved in Walter Deuber and Neil Hindman, *Partitions and sums of* (m,p,c)*-sets*, Journal of Combinatorial Theory A, 1987.

8. Things for future use

??? Not clear! We cannot intersect infinitely many sets in the uf!

$\mathsf{S}_1(\mathcal{S},\mathcal{S})+\mathsf{Split}(\mathcal{S},\mathcal{S})\Rightarrow\mathcal{S}\to(\mathcal{S})^2_2$!. Uses the exercise idea with the $F_n\notin p$.

In fact this gives more: That there are disjoint elements $A_1,A_2\dots\in\mathcal{S}$ such that all edges among them are of the same color. In the presence of S_1 this clearly implies the partition relation.

???

What about the converse direction? If true then better than suf result, since gives complete characterization.

Same for $\mathsf{S}_{\mathrm{fin}}$.

Since $\mathsf{Split}(AP,AP)$, we obtain the known theorem. (It can also be obtained using ONE$\not\uparrow$ $\mathsf{G}_{\mathrm{fin}}(AP,AP)$, indeed TWO$\uparrow$ (AP,AP).)

A superfilter, $\mathsf{Split}(A,B)$, $\mathsf{S}_1(B,C)\Rightarrow A\to(C)^2_2$.

$A>B>C$, so we have $\mathsf{Split}(A,C)$ and $\mathsf{S}_1(A,C)$ but not $\mathsf{S}_1(C,C)$ so unclear if latter can be deduced from a simpler assertion.

Example: $\mathsf{S}_1(O,O)+\mathsf{Split}(\Omega,O)\Rightarrow\Omega\to(O)^2_2$. Easy: $\mathsf{S}_1(O,O)\Rightarrow\mathsf{Split}(\Omega,O)$. We obtain a simpler proof than that in *Open covers and partition relations* (game free), and probably we do not need to make assumptions on the space.

Seems that $Sfin(Om,Om)$ implies Split(Om,Om) via games. NO: u¡d is consistent.

$(\mathsf{Split}(\mathcal{S},\mathcal{S})=CDR(\mathcal{S},\mathcal{S}).)$

CHAPTER 10

Selection principles

Let $(S, \vee)$ be a join semilattice. For elements $a, b \in S$, $a \leq b$ if $a \vee b = b$.

Definition 0.1. Let S be a semilattice. A set $A \subseteq S$ has *no finite subcover* if, for each finite set $F \subseteq A$, there is an element $a \in \mathcal{A}$ with $a \nleq \bigvee F$.

Let $\mathcal{A}$ be a family of countable subsets of S. $\mathcal{A}^{\#}$ is the family of all sets $A \in \mathcal{A}$ with no finite subcover. The family $\mathcal{A}$ is *Menger* if:

(1) If a countable set $B \subseteq S$ has a refinement in $\mathcal{A}$, then $B \in \mathcal{A}$.
(2) For each sequence $A_1, A_2, \dots \in \mathcal{A}^{\#}$, there are finite sets $F_1 \subseteq A_1, F_2 \subseteq A_2, \dots$ with $\bigcup_n F_n \in \mathcal{A}$.

Example 0.2. A topological space X is a Menger space if and only if the family of all countable open covers of X is a Menger family.

We restrict to countable covers, to obtain more general results.

For a set $T \subseteq S$, let $\langle T \rangle$ be the subsemilattice of S generated by T. The following observation is Lemma 3(3) in [Sch06].

Lemma 0.3. *Let $\mathcal{A}$ be a Menger family in a semilattice. Then $\{\, \langle A \rangle : A \in \mathcal{A}^{\#} \,\} \subseteq \mathrm{Mono}(\mathcal{A}) \subseteq \mathcal{A}$.*

Proof. Assume that $\langle A \rangle = A_1 \cup \dots \cup A_k$, and the set A does not refine any set A_i. For each i, pick $a_i \in A$ with $a_i \nleq a$ for all $a \in A_i$. Then $a := a_1 \vee \dots \vee a_k \in \langle A \rangle$, and $a \notin A_i$ for all i; a contradiction. □

If $\mathcal{A}$ is a Menger family and $A \in \mathcal{A}$, then every cofinite subset of A is in $\mathcal{A}$.

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A *union subsystem* of a sequence $U_1, U_2, \dots$ of sets is a sequence

Theorem 0.4. *Let (X, τ) be a Menger space, and let $\{U_1, U_2, \dots\}$ be a large open cover of X with no finite subcover. For each finite coloring of $[\tau]^2$, there are nonempty finite sets $F_1 < F_2 < \cdots$ such that:*

(1) Nice to have: The elements U_F are distinct for distinct F. At least, U_{F_n} are distinct.
(2) *$\{U_{F_1}, U_{F_2}, \dots\}$ is a large cover of X.*
(3) *For all finite sets $H_1 < H_2$, the edges $\{U_{F_{H_1}}, U_{F_{H_1}}\}$ have the same color.*

Proof. Let $(S, \cup)$ be the subsemigroup of $(P(X), \cup)$ generated by the family $\{U_1, U_2, \dots\}$. Let $\mathcal{S}$ be the superfilter consisting of all finite-open covers of X contained in $\mathrm{FS}(U_1, U_2, \dots)$. Let $L = \{\, p \in \beta S : p \subseteq \mathcal{S} \,\}$. Then L is a closed subset of βS.

Since the family $\{U_1, U_2, \dots\}$ is a large cover of X, we have that $\mathrm{FS}(U_n, U_{n+1}, \dots) \in \mathcal{S}$ for all n. Thus, $[\mathrm{FS}(U_n, U_{n+1}, \dots)] \cap L \neq \emptyset$ for all n. Since these are compact sets, the intersection

$$T := \bigcap_n [\mathrm{FS}(U_n, U_{n+1}, \dots)] \cap L$$

is nonempty.

It is known, and easy to see, that the set $\bigcap_n[\mathrm{FS}(U_n, U_{n+1}, \dots)]$ is a subsemigroup of βS. The set L is also a subsemigroup of βS. Indeed, it is a left ideal of βS: Let $s \in S$ and $p \in L$. Let $A \in s+p$. Fix $B \in p$ such that $s+B \subseteq A$. Since B is a finite-open cover of X, so is $s+B$ and therefore so is A. It follows that $S+L \subseteq L$. Since L is a closed subset of βS, we have by right continuity that $\beta S + L \subseteq L$. Thus, the set T is a closed subsemigroup of βS. Let $e \in T$ be an idempotent element.

Fix a set $G \in e$ such that, for each $a \in G$, the set

$$N(a) := \{\, b \in S \setminus \{a\} : \{a, b\} \text{ is green} \,\}$$

is in e. Let $A_1 := G$. For each $n = 1, 2, \dots$:

(1) Let $A_n^* = \{\, a \in A_n : \exists B \in e, a + B \subseteq A_n \,\}$.
(2) Let $A_n' = A_n^* \cap \mathrm{FS}(U_{m_n}, U_{m_n+1}, \dots)$.
(3) Let

Wish Bob to pick just one element. For this, need $G1(Increasing, \Lambda) = Gfin(Omega, \Lambda)$. Should be easy: Modify Bob's moves to their union. Using $Gfin(Omega, \Lambda) = Gfin(Omega, O)$ this is easy. $\square$

Corollary 0.5. *One-dim case.*