WAS GÖDEL RIGHT?

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Set theory began with Cantor's work, when he was studying some special sets of real numbers, the accumulation points of a given set of reals.

This study led Cantor to the following fundamental question: does there exist a bijection between the set of natural numbers and the set of real numbers?

Cantor answered this question negatively by showing that there is no such function. Cantor's theorem is more general and actually he showed that there is no surjection from a set X to the set of the subsets of X.

Cantor's work did not stop here, and with his sharp intuition he discovered new concepts like the aleph's scale:

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \aleph_{\omega+1} \\ \aleph_\omega \\ \bullet \\ \bullet \\ \aleph_1 \\ \aleph_0 \\ \bullet \\ \bullet \\ 1 \\ 0 \end{array}$$

Using this notation, we have that the cardinality of the continuum is the cardinality of $2^{\aleph_0}$, the set of all subsets of natural numbers. Therefore, Cantor's theorem says

$$\aleph_0 < 2^{\aleph_0}$$

and Cantor's question was

$$\text{IS } 2^{\aleph_0} \text{ EQUAL TO } \aleph_1?$$

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A real advance on Cantor's question was given by Gödel when he showed "If the usual axioms for set theory are consistent, then there is a model for these axioms where the continuum has cardinality $\aleph_1$".

This essential contribution to Cantor's problem was made by Gödel in his work concerning the constructible universe and was stated in 1936 [GO].

At the end of the 50's the following contribution was given by Hajnal and Levy: if there is a model for the negation of CH then there is one of the form $L[A]$ [HA; LE].

Between Gödel's work and Cohen's 1964 paper no real advance on Cantor's problem was made. The only interesting fact was a work of Gödel where he explained the mathematical concepts involved in Cantor's problem. Many people say that Gödel's opinion was that the continuum cardinality is in fact $\aleph_2$.

In 1963, Paul Cohen [CO] showed that if the ZF-axioms for set theory are consistent, then there is a model for set theory where the continuum is bigger than $\aleph_1$.

Actually Cohen's method showed that the only restriction for the cardinality of the continuum is König's lemma, which says that its cofinality must be bigger than ω.

I see Cohen's work as a turning point in the history of set theory: This is the end of classic set theory and the beginning of a new era. What Cohen did for set theory is similar to what Galois did for algebra. However, the real transformation of the "old set theory" was given by Solovay's work. In one paper, Solovay showed that there is a model where "every projective set of reals is measurable" [SO], and in another paper he and Tennebaum showed the consistency of Martin's axiom with $\neg CH$ [ST]. In a third work with D. Martin, they did a complete analysis of Martin's axiom [MS]. In these three papers, we can find the basic elements used in the post-Cohen era of set theory. Looking at the first two papers, we can see that two different forcing techniques are used, namely, product of forcing and iterations of forcing. It is not hard to see that the second one is the more elaborate technique, and it is being used successfully to solve many problems in set theory.

In time, this technique came to be called "finite support iteration", and the first failure of this technique occurred when it was discovered that the iteration, containing cofinally many forcing notions not satisfying the countable chain condition, collapses $\aleph_1$.

The second failure of this technique is that in every ω-stage, we are adding Cohen reals over the ground model.

Why is this bad?

There are two answers to this question, namely

(1) Because when we are using a technique, we want to have complete control of its actions.

(2) Because there are many problems which are incompatible with models which are full of Cohen reals.

The technique used to get around this problem was introduced by R. Laver in his paper "On the consistency of Borel's conjecture". In his paper, Laver introduced the concept of countable support iteration. Using this technique [LA], people solved a large number of problems. S. Shelah [SH] gave a systematic study of countable support iteration and he discovered the very important concepts of proper forcing and semi-proper forcing. All of Shelah's technology was developed in order to find that some property, say PROPER(TY), and a framework for iteration satisfying

(1) the composition of two forcing notions satisfying PROPER(TY) satisfies PROPER(TY);
(2) the iteration, using the framework of forcing, of forcing notions satisfying PROPER(TY), satisfies PROPER(TY);
(3) if a forcing notion satisfies PROPER(TY), then $\aleph_1$ is the same in the ground model and in the generic extention.

The outcome of Shelah's program is well known.

Unfortunately, countable support iteration has a problem that was pointed out by many people, including J. Roitman, and the problem is: There is no way to make $2^{\aleph_0} = \aleph_3 (\aleph_4 \dots)$ using countable support iteration [BA].

In the beginning people expressed the failure of this technique by asking:

Is Borel's conjecture consistent with $2^{\aleph_0} > \aleph_2$?

People thought that the solution of this problem was instrinsically connected with some new framework for iterated forcing. Unfortunately, this problem was not the right one, as [ISW] showed. (It is enough to add Solovay reals to the Laver model in order to enlarge the continuum and preserve the Borel conjecture.) However, the fact that "the Borel conjecture problem" was not the right one, does not say that there is no right problem. On the contrary, we feel that we are close to an intrinsic problem about the size of the continuum. The problem at this moment is to find a simple question whose solution will shed light on the iteration problem.

There are previous works in this direction. For instance, in a paper of Groszek and Jech, "Generalized iteration of forcing," they found a new style of iteration, but unfortunately again, by adding Cohen reals and making the universe flat but not high. The main result of this work has been improved, recently, by B. Velickovic using the old technique of finite support iteration.

There is an effort by Shelah using some kind of "three coordinate", but this iteration uses intermediate stages satisfying Martin's axiom. In [ISW], we introduce an iteration framework; but unfortunately, this only has nice property if in the intermediate stage there are Ramsey ultrafilters, but these objects are not guaranteed by ZFC. However, this approach gives a c.c.c. (σ-centered) poset forcing Borel's conjecture.

We were very optimistic about this project, but Canjal has shown that

"Every filter of size less than the continuum can be extended to a Ramsey ultrafilter iff the real line cannot be covered by less than c many meager sets".

Now it is interesting to see what is the status of some problems that had been handled by set theorists. These problems are completely open when the continuum is $> \aleph_2$.

Let me introduce some definitions: A filter F on ω is called p-point if $\forall \langle a_n : n < \omega \rangle$, $a_n \in F$, there is $a \in F$ such that

$$\|a - a_n\| < \aleph_0 \text{ for all } n.$$

The existence of this object in $\beta\omega$ is of interest to analysts.

It is not hard to see that

$$ZFC + CH \;\vdash\; \text{"there are } p\text{-points"}$$

Shelah [SH] showed that there is M such that

$$M \models \text{"}ZFC + 2^{\aleph_0} = \aleph_2\text{"}$$

$$M \;\models\; \text{"there is no } p\text{-point"}$$

It is completely open whether there is such an M satisfying $2^{\aleph_0} > \aleph_2$. (Shelah claims that he has a project for this).

Cantor showed that every two countable dense linear order, without end-points, are isomorphic. It is very natural to ask the same for $\aleph_1$-dense linear ordering. In this direction there are three papers giving a complete solution to this problem. (Baumgartner; Abraham–Shelah; Abraham–Rubin–Shelah):

There are models where they are isomorphic and models where they are not isomorphic, including models satisfying MA. It is completely open what happens with the $\aleph_2$-dense linear orderings.

Let X be a set of reals. We say that X can be mapped continuously onto the reals if there exists a continuous function $\Im$ such that range $(\Im \restriction X) = \mathbb{R}$. It is clear that universal measure zero set cannot be mapped continuously onto the reals, therefore under CH, or weak forms of MA, there are sets of reals which cannot be mapped continuously onto the reals. However, A. Miller had showed that there is a model for $2^{\aleph_0} = \aleph_2$ where every set of reals of cardinality $\aleph_2$ can be mapped continuously onto the reals. Again, it is completely open if this is true when $2^{\aleph_0} > \aleph_2$. This problem have a Turing degree version.

This list of examples could be enlarged very much. There is a large list of problems that are solved when $2^{\aleph_0} = \aleph_2$ and are open for $2^{\aleph_0} > \aleph_2$.

There is one more interesting phenomenon:

$\aleph_2$ is also discriminating against

There are some questions that are decidable under the assumption of $2^{\aleph_0} = \aleph_2$, but are completely open when $2^{\aleph_0} > \aleph_2$. To give some examples let me introduce some notation: We say that $\langle a_i : i < \alpha \rangle$ is a tower if $a_i \in \omega$ and for $i < j < \alpha$, $a_j \subseteq^* a_i (\|a_j - a_i\| < \aleph_0)$. Let t be the minimal cardinal such that for every tower $\langle a_i : i < \alpha \rangle$, $\alpha < t$, there is $a \subseteq \omega$ such that

$$a \subseteq^* a_i \text{ for every } i < \alpha.$$

Let p be the minimal cardinal such that for every filter F on ω, if $|F| < p$ then there is $a \subseteq \omega$ such that

$$a \subseteq^* b \text{ for every } b \in F.$$

ROTHBERGER: $2^{\aleph_0} = \aleph_2 \vdash p = t$
IT IS COMPLETELY OPEN WHETHER

$$ZFC \vdash p = t$$

A q-point is a filter F on ω satisfying: For every $f : \omega \to \omega$, if for every $n \in \omega$ $f^{-1}(n)$ is finite then there is $a \in F$ such that

$$f \restriction a \text{ is one-to-one}$$

(some people call F selective)
KETONE-MATHIAS:

$$2^{\aleph_0} = \aleph_2 \ \vdash \ \text{“There are } p \text{ or } q \text{ points”}$$

THE QUESTION FOR $2^{\aleph_0} > \aleph_2$ IS COMPLETELY OPEN.
Let X be a topological space. We say that X omits the cardinal λ if

$$(\forall Y \subseteq X)(Y \text{ is closed then } \operatorname{card}(Y) \neq \lambda)$$

In [GIS] we proved

$$ZFC \ \vdash \ \text{“there is a regular space omitting } \aleph_2\text{”}$$

It is completely open if

$$ZFC \ \vdash \ \text{“there is a compact space omitting } \aleph_2\text{”}$$

Also, set theorists have found some combinatorial axioms that prove that

$$2^{\aleph_0} = \aleph_2$$

Five years ago, Foreman–Magidor–Shelah found a new axiom called "Martin Maximum" (Shelah showed that is equivalent to semi-proper forcing axiom). This axiom is like Martin's axiom, but it also guarantees generic objects; for posets that preserves stationary subsets of ω_1. They proved

$$\text{Martin Maximum} \vdash 2^{\aleph_0} = \aleph_2.$$

After this result, Todorcevic showed that

$$\text{Proper Forcing Axiom} \vdash \text{"}2^{\aleph_0} = \aleph_2\text{"}$$

Looking at the proof of this theorem, we see that the portion of the proper forcing axiom necessary for this theorem, is very small but still includes large cardinal assumptions.

We can see all the above mentioned results as "accumulated evidence" for the assumption $2^{\aleph_0} = \aleph_2$.

Let me show now that the assumption $2^{\aleph_0} = \aleph_2$ still leaves the set theory of the real line highly non-trivial, by exhibiting some problems that remain open under this assumption.

The first problem involves large cardinal assumptions. If we assume that the set theory universe is closed under Sharp's, then we define for each $x \in \mathbb{R}$

$$C_x = \{\lambda : \lambda \text{ in a cardinal in } L[x]\}$$

Clearly $\{\aleph_1, \aleph_2\} \subseteq \cap\{C_x : x \in \mathbb{R}\} = C$

Call θ the second element of C. Then $\aleph_1 < \theta \leq \aleph_2$. Starting from a model for AD, Steel–Van Wesep–Woodin, showed that there is a model for $\theta = \aleph_2$.

Foreman–Magidor have proved the following result:

If P is proper, then

$$V^P \models \delta_2^1 = (\delta_2^1)^V$$

and this implies that if

$$V \models \theta < \aleph_2$$

then

$$V^P \models \theta < \aleph_2$$

They have the following conjecture:

If $V \models$ "There is a supercompact cardinal", then $V \models$ "$\theta < \aleph_2$".

On the other hand, Solovay conjectured: From some large cardinal assumption, it is possible to get a model M satisfying

$M \models$ "there is a supercompact cardinal"
$M \models$ "$\theta = \aleph_2$".

We think that a construction of a model, for Solovay's conjecture, might require new conditions that must be weaker than semi-proper forcing, but still preserving $\aleph_1$. I know that Shelah has a candidate for this. Woodin thinks that the solution is connected with the inner model problem. Time will say what will be the solution.

The following problem also involves large cardinals: S. Shelah showed that

$$(\aleph_\omega)^\omega < \aleph_c^+$$

Gitik–Woodin showed, from the consistency of a cardinal k of measurability degree $0(k) = k^{++}$, that it is possible to get a model of $(\aleph_\omega)^\omega = \aleph_{\omega+2}$ and GCH below $\aleph_\omega$.

From stronger assumptions, it is possible to get GCH below $\aleph_\omega$ and $(\aleph_\omega)^\omega = \aleph_{\omega+\alpha}$, when $\alpha < \aleph_1$. But it is completely unclear if the Shelah bound can be improved upon, or what large cardinal assumptions are required to get models approximating Shelah's bound.

I don't want to forget the Mathian problem about the consistency strength of "every projective set of reals is Ramsey". I know of at least four good set-theorists who are trying to solve it.

Some time ago it was obvious that category and measurability were symmetrical. For example, I remember that when Shelah solved that Σ^1_3-measurability implies "$\aleph_1$ is inaccessible in L", he said that a symmetric argument would work for category. However, Shelah later showed that in this case measure and category are not symmetric. At this time, we have a large list of asymmetries, but it is not clear what is the exact nature of this asymmetry; for example, in a joint paper with Shelah [IS1], we showed that the asymmetry between category and measure, for projective sets, disappear if the universe satisfies a small part of Martin's axiom. Sierpinski showed that ultrafilers on ω produces non-measurable and non-Baire property sets. We showed that a filter does not have the Baire property iff it is unbounded [IH]. The following question remains open

BARTOSZYNSKI: Does ZFC imply that there is a bounded non-measurable filter on ω?

Finally, to end this list of problems exemplifying that the assumption of $2^{\aleph_0} = \aleph_2$ does not trivialize the set theory of the real line, I want to explain why Martin's axiom does not completely satisfy my taste. The problem is that

$$MA \vdash 2^{\aleph_0} = 2^{\aleph_1}$$

This consequence of MA is, from my point of view, very strange, and we think that it should be very interesting to find the largest family of poset, say F, such that there is a model

$$M \models \text{“}MA(F) + 2^{\aleph_0} < 2^{\aleph_1}\text{”}$$

A family like this was defined in a joint work with Shelah [IS2], called "Souslin forcing." But it is not the largest one.

CONCLUSIONS.

It was clear that Cantor's question was not solved by Gödel–Cohen works. However, people thought that, working in ZFC and using forcing, we shall never be able to get some intuition to the real size of the continuum. Nevertheless, today, after hard technical work, the facts show us that it is possible to find some kind of evidence about the continuum size. If it is $\aleph_2$ we don't know yet, but a theorem proving the impossibility of enlarging the continuum to $\aleph_3$, in a general way, will be a strong argument for Gödel's intuition.

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