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On the cardinality of the θ-closed hull of sets

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ABSTRACT

The *θ-closed hull* of a set A in a topological space is the smallest set C containing A such that, whenever all *closed* neighborhoods of a point intersect C, this point is in C.
We define a new topological cardinal invariant function, the *θ-bitightness small number* of a space X, $\mathrm{bts}_\theta(X)$, and prove that in every topological space X, the cardinality of the θ-closed hull of each set A is at most $|A|^{\mathrm{bts}_\theta(X)}$. Using this result, we synthesize all earlier results on bounds on the cardinality of θ-closed hulls. We provide applications to P-spaces and to the almost-Lindelöf number.

1. Introduction

An *Urysohn* (or $T_{2\frac{1}{2}}$) *space*, is a space in which distinct points are separated by closed neighborhoods. Thus, Urysohn spaces are in between Hausdorff and regular spaces. The spaces considered here generalize Urysohn spaces.

Let X be a topological space. A point $x \in X$ is in the *θ-derivative* $\theta(A)$ of a set $A \subseteq X$ if each closed neighborhood of x intersects A (cf. Veličko [11]).[1] For regular spaces, $\theta(A) = \overline{A}$, but in general the operator

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[1] The use of the letter θ for these concepts was proposed by Alexandroff, in recognition of Fedorchuk's results on the involved concepts (θ is the first letter in a Greek transcription of "Fedorchuk".) For additional details on the history of and motivation for the concepts treated in this paper, see [7].

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θ is not idempotent for Urysohn spaces.[2] The *θ-closed hull* $\overline{A}^{\theta}$ of A (cf. [3]) is the smallest set $C \subseteq X$ such that $A \subseteq C = \theta(C)$.[3]

As there are first countable Urysohn spaces X and sets $A \subseteq X$ such that, e.g., $|\overline{A}| = \aleph_0 < 2^{\aleph_0} = \theta(A)$ [3], a major goal concerning the mentioned concepts is that of providing upper bounds on the cardinalities of θ-closed hulls of sets, in terms of cardinal functions of the ambient space X (e.g., Bella and Cammaroto [3], Cammaroto and Kočinac [8,9], Bella [2], Alas and Kočinac [1], Bonanzinga, Cammaroto and Matveev [5], Bonanzinga and Pansera [6], and McNeill [10]). We identify several concepts and topological cardinal functions, which lead to generalizations of results from the mentioned papers.

Throughout this paper, X is a topological space and A is an arbitrary subset of X.

Recall that for $x \in X$, $\chi(X,x)$ is the minimal cardinality of a local base at x, and the *character* $\chi(X)$ of X is the maximum of $\aleph_0$ and $\sup_{x\in X} \chi(X,x)$. In 1988, Bella and Cammaroto proved that, for Urysohn spaces X, $|\overline{A}^{\theta}| \leqslant |A|^{\chi(X)}$ [3].

For $x \in X$, let $\chi_\theta(X,x)$ be the minimal cardinality of a family of *closed* neighborhoods of x such that each closed neighborhood of x contains one from this family. The *θ-character* $\chi_\theta(X)$ of X is the maximum of $\aleph_0$ and $\sup_{x\in X} \chi_\theta(X,x)$. Thus, $\chi_\theta(X) \leqslant \chi(X)$. In [1], Alas and Kočinac define this topological cardinal invariant, show that the inequality may be proper, and modify the Bella–Cammaroto argument to show that, for Urysohn spaces X, $|\overline{A}^{\theta}| \leqslant |A|^{\chi_\theta(X)}$.

In 1993, Cammaroto and Kočinac defined the *θ-bitightness* of an Urysohn space X, $\mathrm{bt}_\theta(X)$, to be the minimal cardinal κ such that, for each non-θ-closed $A \subseteq X$, there are $x \in \theta(A) \setminus A$ and sets $A_\alpha \in [A]^{\leqslant\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) = \{x\}$ [8]. For Urysohn spaces X, Cammaroto and Kočinac proved that $\mathrm{bt}_\theta(X) \leqslant \chi(X)$. Moreover, their proof shows that $\mathrm{bt}_\theta(X) \leqslant \chi_\theta(X)$. They supplied examples where the inequality is strict, and proved that $|\overline{A}^{\theta}| \leqslant |A|^{\mathrm{bt}_\theta(X)}$, thus refining the Bella–Cammaroto Theorem.

In their recent work [5], Bonanzinga, Cammaroto and Matveev defined the *Urysohn number* $\mathrm{U}(X)$ to be the minimal cardinal κ such that, for each set $\{x_\alpha\colon \alpha < \kappa\} \subseteq X$, there are closed neighborhoods U_α of x_α, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} U_\alpha = \emptyset$. Thus, X is Urysohn if and only if $\mathrm{U}(X) = 2$. They note that, for Hausdorff spaces, $\mathrm{U}(X) \leqslant |X|$, and prove that for each cardinal $\kappa \geqslant 2$, there is a Hausdorff space with $\mathrm{U}(X) = \kappa$ [5].

Definition 1.1. X is *finitely-Urysohn* if $\mathrm{U}(X)$ is finite.

Bonanzinga, Cammaroto and Matveev generalized the result by Bella and Cammaroto from Urysohn to finitely-Urysohn spaces [5]. Later, Bonanzinga and Pansera improved this and the result by Alas and Kočinac: For finitely-Urysohn spaces, $|\overline{A}^{\theta}| \leqslant |A|^{\chi_\theta(X)}$ [6].

A technical problem in synthesizing the Bonanzinga–Cammaroto–Matveev theorem and the Cammaroto–Kočinac theorem is that $\mathrm{bt}_\theta(X)$ need not be defined for finitely-Urysohn spaces.

We define a new topological cardinal invariant function, the *θ-bitightness small number* of a space X, denoted $\mathrm{bts}_\theta(X)$, and prove the following assertions:

(1) $\mathrm{bts}_\theta(X)$ is defined for all topological spaces X (Definition 2.7).
(2) Whenever $\mathrm{bt}_\theta(X)$ is defined, $\mathrm{bts}_\theta(X) \leqslant \mathrm{bt}_\theta(X)$ (Corollary 2.2 and Definition 2.7).
(3) For all finitely-Urysohn spaces, $\mathrm{bts}_\theta(X) \leqslant \chi_\theta(X)$ (Theorem 2.6 and Definition 2.7).
(4) In every topological space X, $|\overline{A}^{\theta}| \leqslant |A|^{\mathrm{bts}_\theta(X)}$ (Theorem 2.8).

This generalizes all of the above-mentioned results. The situation is summarized in the following diagram.

[2] In earlier works, the θ-derivative $\theta(A)$ is also denoted $\mathrm{cl}_\theta(A)$ and called *θ-closure*. Since the operator θ is not idempotent, we decided not to use the term *closure* here.

[3] In earlier works, the θ-closed hull of A is also denoted $[A]_\theta$.

$$\begin{array}{ccc}
\forall \text{ finitely-Urysohn } X,\ |\overline{A}^{\theta}| \leqslant |A|^{\chi(X)} \text{ [5]} & \longrightarrow & \forall \text{ Urysohn } X,\ |\overline{A}^{\theta}| \leqslant |A|^{\chi(X)} \text{ [3]} \\
\uparrow & & \uparrow \\
\forall \text{ finitely-Urysohn } X,\ |\overline{A}^{\theta}| \leqslant |A|^{\chi_\theta(X)} \text{ [6]} & \longrightarrow & \forall \text{ Urysohn } X,\ |\overline{A}^{\theta}| \leqslant |A|^{\chi_\theta(X)} \text{ [1]} \\
\uparrow & & \uparrow \\
\forall X,\ |\overline{A}^{\theta}| \leqslant |A|^{\mathrm{bts}_\theta(X)} & \longrightarrow & \forall \text{ Urysohn } X,\ |\overline{A}^{\theta}| \leqslant |A|^{\mathrm{bt}_\theta(X)} \text{ [8]}
\end{array}$$

We actually establish finer theorems than the ones mentioned above, as explained in the following sections.

We also provide a partial solution to a problem of Bonanzinga, Cammaroto and Matveev [5] and Bonanzinga and Pansera [6].

2. Finite bitightness and the bitightness small number

Definition 2.1. The *finite θ-bitightness* of a space X, $\mathrm{fbt}_\theta(X)$, is the minimal cardinal κ such that, for each non-θ-closed $A \subseteq X$, there are sets $A_\alpha \in [A]^{\leqslant\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) \setminus A$ is finite and nonempty.

Corollary 2.2. *$\mathrm{fbt}_\theta(X)$ is defined for all finitely-Urysohn spaces. When $\mathrm{bt}_\theta(X)$ is defined, so is $\mathrm{fbt}_\theta(X)$, and $\mathrm{fbt}_\theta(X) \leqslant \mathrm{bt}_\theta(X)$.*

The following easy fact will be used in several occasions.

Lemma 2.3. *If $x \in \theta(A)$, then for each closed neighborhood V of x, $x \in \theta(A \cap V)$.*

For Urysohn spaces, $\mathrm{fbt}_\theta(X)$ is very closely related to $\mathrm{bt}_\theta(X)$.

Proposition 2.4. *Let X be an Urysohn space, and $\kappa = \mathrm{fbt}_\theta(X)$. For each non-$\theta$-closed $A \subseteq X$, there are $x \notin A$ and $A_\alpha \in [A]^{\leqslant\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) \setminus A = \{x\}$.*

Proof. Pick sets $A_\alpha \in [A]^{\leqslant\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) \setminus A$ is finite, say equal to $\{x_1, \dots, x_k\}$.

Since X is Urysohn, there are closed neighborhoods V_i of x_i, $i \leqslant k$, such that $V_1 \cap (V_2 \cup \cdots \cup V_k) = \emptyset$. Indeed, for each $i = 2, \dots, k$ pick disjoint closed neighborhoods U_i and V_i of x_1, x_i, respectively, and set $V_1 = U_2 \cap \cdots \cap U_k$.

For each $\alpha < \kappa$, $x_1 \in \theta(A_\alpha \cap V_1)$. Then $A_\alpha \cap V_1 \in [A]^{\leqslant\kappa}$ for each α, and

$$\bigcap_{\alpha<\kappa} \theta(A_\alpha \cap V_1) \setminus A = \{x_1\}. \qquad \square$$

Lemma 2.5. *Let X be a finitely-Urysohn space. For all $B, D \subseteq X$ with $B \subseteq \theta(D)$ and $|B| \geqslant \mathrm{U}(X)$, there are $1 \leqslant m \leqslant k < \mathrm{U}(X)$ and $b_1, \dots, b_k \in B$ such that*

$$B \cap \bigcap_{V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_k)} \theta(D \cap V) = \{b_1, \dots, b_m\}. \tag{1}$$

Proof. For $k = \mathrm{U}(X)$, any intersection as in (1) is empty. For $k = 1$, any such intersection is nonempty (since, by Lemma 2.3, it contains b_1). Thus, let k be maximal such that there are $b_1, \dots, b_k \in B$ for which the intersection in (1) is nonempty. $1 \leqslant k < \mathrm{U}(X)$. We claim that

$$B \cap \bigcap_{V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_k)} \theta(D \cap V) \subseteq \{b_1, \dots, b_k\}.$$

Assume, towards a contradiction, that there is

$$x \in B \cap \bigcap_{V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_k)} \theta(D \cap V) \setminus \{b_1, \ldots, b_k\}.$$

By Lemma 2.3, for each $V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_k)$ and each $W \in \mathcal{N}_\theta(x)$, $x \in \theta(D \cap V \cap W)$. Thus,

$$x \in B \cap \bigcap_{V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_k) \wedge \mathcal{N}_\theta(x)} \theta(D \cap V),$$

and in particular this set is nonempty. This contradicts the maximality of k.

Thus, the intersection is nonempty, and by reordering $b_1, \ldots, b_k$, we may assume that the intersection is $\{b_1, \ldots, b_k\}$ for some m with $1 \leqslant m \leqslant k$. □

Theorem 2.6. *For each finitely-Urysohn space X, $\mathrm{fbt}_\theta(X) \leqslant \chi_\theta(X)$.*

Proof. For families of sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n \subseteq P(X)$, define

$$\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \cdots \wedge \mathcal{F}_n := \{V_1 \cap V_2 \cap \cdots \cap V_n\colon\ V_1 \in \mathcal{F}_1, \ldots, V_n \in \mathcal{F}_n\}.$$

For $x \in X$, let $\mathcal{N}_\theta(x)$ be the family of closed neighborhoods of x.

Let $\kappa = \chi_\theta(X)$. Let $A \subseteq X$ be non-θ-closed. Assume that $\theta(A) \setminus A$ is finite. Fix $b \in \theta(A) \setminus A$. Fix a base $\{V_\alpha\colon\ \alpha < \kappa\}$ for $\mathcal{N}_\theta(b)$. For each $\alpha < \kappa$, let $a_\alpha \in A \cap V_\alpha$. Let $D = \{a_\alpha\colon\ \alpha < \kappa\}$, and set $A_\alpha = D$ for all $\alpha < \kappa$. Then

$$b \in \theta(D) \setminus A = \bigcap_{\alpha < \kappa} \theta(A_\alpha) \setminus A \subseteq \theta(A) \setminus A,$$

so that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) \setminus A$ is finite and nonempty, and the requirement in the definition of $\mathrm{fbt}_\theta(X) \leqslant \kappa$ is fulfilled.

Thus, assume that the set $B = \theta(A) \setminus A$ is infinite. Apply Lemma 2.5 to the sets B and $D = A$, to obtain $1 \leqslant m \leqslant k < \mathrm{U}(X)$ and $b_1, \ldots, b_k \in B$ such that Eq. (1) holds. For each $i \leqslant k$, fix a basis $\mathcal{F}_i$ for $\mathcal{N}_\theta(b_i)$ with $|\mathcal{F}_i| \leqslant \kappa$. Enumerate

$$\mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_k = \{V_\alpha\colon\ \alpha < \kappa\}.$$

By Eq. (1),

$$B \cap \bigcap_{\alpha < \kappa} \theta(A \cap V_\alpha) = \{b_1, \ldots, b_m\}.$$

In particular, for each $\alpha < \kappa$ there is $a_\alpha \in A \cap V_\alpha$. Take

$$C = \{a_\alpha\colon\ \alpha < \kappa\} \in [A]^{\leqslant\kappa}.$$

Fix $i \leqslant m$ and $\alpha < \kappa$. Let $V \in \mathcal{N}_\theta(b_i)$. Then $V_\alpha \cap V \in \mathcal{N}_\theta(b_1) \wedge \cdots \wedge \mathcal{N}_\theta(b_m)$, and thus there is $\beta < \kappa$ such that $V_\beta \subseteq V_\alpha \cap V$. Then $a_\beta \in C \cap V_\alpha \cap V$, and in particular $C \cap V_\alpha \cap V$ is nonempty. This shows that $b_i \in \theta(C \cap V_\alpha)$.

Thus,

$$b_1,\dots,b_m \in \bigcap_{\alpha<\kappa}\theta(C\cap V_\alpha)\setminus A \subseteq \bigcap_{\alpha<\kappa}\theta(A\cap V_\alpha)\setminus A$$
$$\subseteq \bigl(\theta(A)\setminus A\bigr)\cap\bigcap_{\alpha<\kappa}\theta(A\cap V_\alpha)=\{b_1,\dots,b_m\},$$

and therefore

$$\bigcap_{\alpha<\kappa}\theta(C\cap V_\alpha)\setminus A=\{b_1,\dots,b_m\},$$

as required in the definition of $\mathrm{fbt}_\theta(X)\leqslant\kappa$. □

Definition 2.7. The *θ-bitightness small number* of X, $\mathrm{bts}_\theta(X)$, is the minimal cardinal κ such that, for each non-θ-closed $A\subseteq X$ that is not a singleton,[4] there are $A_\alpha\in[A]^{\leqslant\kappa}$, $\alpha<\kappa$, such that

$$\bigcap_{\alpha<\kappa}\theta(A_\alpha)\setminus A\neq\emptyset\quad\text{and}\quad\left|\bigcap_{\alpha<\kappa}\theta(A_\alpha)\right|\leqslant|A|^\kappa.$$

$\mathrm{bts}_\theta(X)$ is defined for all spaces X, and is obviously $\leqslant\mathrm{fbt}_\theta(X)$ whenever the latter is defined.

Theorem 2.8. *Let X be a topological space. For each $A\subseteq X$,*

$$\left|\overline{A}^\theta\right|\leqslant|A|^{\mathrm{bts}_\theta(X)}.$$

Proof. Let $\kappa=\mathrm{bts}_\theta(X)$, $\lambda=|A|$. We define sets $C_\alpha\subseteq X$, with $|C_\alpha|\leqslant\lambda^\kappa$, $\alpha\leqslant\kappa^+$, by induction on α.

$C_0:=A$.

Given C_α,

$$C_{\alpha+1}:=\bigcup\left\{\bigcap_{\beta<\kappa}\theta(B_\beta)\colon\ \{B_\beta\colon\ \beta<\kappa\}\subseteq[C_\alpha]^{\leqslant\kappa},\ \left|\bigcap_{\beta<\kappa}\theta(B_\beta)\right|\leqslant\lambda^\kappa\right\}.$$

Then $C_\alpha\subseteq C_{\alpha+1}$. As $|C_\alpha|\leqslant\lambda^\kappa$, $|C_{\alpha+1}|\leqslant((\lambda^\kappa)^\kappa)^\kappa\cdot(\lambda^\kappa)^\kappa=\lambda^\kappa$.

For a limit ordinal α, $C_\alpha:=\bigcup_{\beta<\alpha}C_\beta$. Then $|C_\alpha|\leqslant|\alpha|\cdot\lambda^\kappa\leqslant\kappa^+\cdot\lambda^\kappa=\lambda^\kappa$.

End of the construction.

Let $C=C_{\kappa^+}$. Then $|C|\leqslant\lambda^\kappa$, $A=C_0\subseteq C$, and C is θ-closed. Indeed, assume otherwise and let $B_\alpha\in[C]^{\leqslant\kappa}$, $\alpha<\kappa$, be such that $\bigcap_{\alpha<\kappa}\theta(B_\alpha)\setminus C\neq\emptyset$ and $|\bigcap_{\alpha<\kappa}\theta(B_\alpha)|\leqslant|C|^\kappa$. Then $|\bigcap_{\alpha<\kappa}\theta(B_\alpha)|\leqslant(\lambda^\kappa)^\kappa=\lambda^\kappa$. As κ^+ is regular, for each $\alpha<\kappa$ there is $\beta_\alpha<\kappa^+$ such that $B_\alpha\subseteq C_{\beta_\alpha}$. Again as κ^+ is regular, $\beta:=\sup_{\alpha<\kappa}\beta_\alpha<\kappa$. Then $B_\alpha\in[C_\beta]^{\leqslant\kappa}$ for all $\alpha<\kappa$, and thus $\bigcap_{\alpha<\kappa}\theta(B_\alpha)\subseteq C_{\beta+1}\subseteq C$. A contradiction. □

Remark 2.9. Immediately after Proposition 7 of [2], Bella points out that there are Hausdorff spaces X where the inequality $|\overline{A}^\theta|\leqslant|A|^{\chi(X)}$ fails for some of their subsets. In particular, by Theorem 2.8, $\mathrm{bts}_\theta(X)$ may be larger than $\chi_\theta(X)$ may fail for general Hausdorff spaces X.

[4] In the Hausdorff context, singletons are θ-closed, and thus the restriction to non-singletons may be removed.

3. The θ-closed hull in P-spaces

Bonanzinga, Cammaroto and Matveev [5] and Bonanzinga and Pansera [6] ask whether, in all Hausdorff spaces X, $|\overline{A}^\theta| \leqslant |A|^{\chi_\theta(X)} \cdot \mathrm{U}(X)$. We give a partial answer.

Definition 3.1. The *θ-P-point number* of a space is the minimal cardinal κ such that some $x \in X$ has closed neighborhoods V_α, $\alpha < \kappa$, with $\bigcap_{\alpha<\kappa} V_\alpha$ not a neighborhood of x.

As the *θ-P-point number* of any space is at least $\aleph_0$, the following theorem generalizes the Bonanzinga–Pansera Theorem, and thus also the earlier three theorems discussed in the introduction.

Theorem 3.2. *Let X be a topological space whose Urysohn number is smaller than its θ-P-point number. For each $A \subseteq X$,*

$$\left|\overline{A}^\theta\right| \leqslant |A|^{\chi_\theta(X)} \cdot \mathrm{U}(X).$$

Proof. Let $\kappa = \chi_\theta(X)$. For each $x \in \theta(A)$, let $\{V^x_\alpha\colon\ \alpha < \kappa\}$ be a family of closed neighborhoods of x such that each closed neighborhood of x contains one from this family. For each $\alpha < \kappa$, fix $a_{x,\alpha} \in A \cap V^x_\alpha$. Let $A_x = \{a_{x,\alpha}\colon\ \alpha < \kappa\}$.

Define a map

$$\begin{aligned} \Psi : \theta(A) &\to \left[[A]^{\leqslant\kappa}\right]^{\leqslant\kappa}, \\ x &\mapsto \left\{A_x \cap V^x_\alpha\colon\ \alpha < \kappa\right\}. \end{aligned}$$

Let $\nu = \mathrm{U}(X)$. Let x_α, $\alpha < \nu$, be distinct elements of $\theta(A)$ which are all mapped to the same element $\Psi(x)$. For each $\alpha < \nu$, pick $\beta_\alpha < \kappa$ such that

$$\bigcap_{\alpha<\nu} V^{x_\alpha}_{\beta_\alpha} = \emptyset.$$

Let $\alpha < \nu$. As $\Psi(x_\alpha) = \Psi(x)$, there is $\gamma_\alpha < \kappa$ such that $A_{x_\alpha} \cap V^{x_\alpha}_{\beta_\alpha} = A_x \cap V^x_{\gamma_\alpha}$. As ν is smaller than the θ-P-point number of X, $V := \bigcap_{\alpha<\nu} V^x_{\gamma_\alpha}$ is a closed neighborhood of x. Fix $\delta < \kappa$ such that $V^x_\delta \subseteq V$. Then

$$\begin{aligned} a_{x,\delta} \in A_x \cap V^x_\delta \subseteq A_x \cap V = A_x \cap \bigcap_{\alpha<\nu} V^x_{\gamma_\alpha} &= \bigcap_{\alpha<\nu} A_x \cap V^x_{\gamma_\alpha} \\ = \bigcap_{\alpha<\nu} A_{x_\alpha} \cap V^{x_\alpha}_{\beta_\alpha} \subseteq \bigcap_{\alpha<\nu} V^x_{\beta_\alpha} &= \emptyset; \end{aligned}$$

a contradiction.

Thus, Ψ is $< \nu$ to 1, and therefore the cardinality of $\overline{A}^\theta$ is at most

$$\left|\left[[A]^{\leqslant\kappa}\right]^{\leqslant\kappa}\right| \cdot \nu = |A|^\kappa \cdot \nu.$$

By induction on $\alpha \leqslant \kappa^+$, define $A_0 := A$, $A_{\alpha+1} := \theta(A_\alpha)$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for limit ordinals α. Then, by induction, $|A_\alpha| \leqslant |A|^\kappa \cdot \nu$ for all α. As $\chi_\theta(X) = \kappa$, $A_{\kappa^+} = \overline{A}^\theta$ [6]. □

Recall that X is a *P-space* if each countable intersection of neighborhoods is a neighborhood. Thus, the θ-P-point number of a P-space is $\geqslant \aleph_1$.

Corollary 3.3. *Let X be a P-space with $\mathrm{U}(X) = \aleph_0$. For each $A \subseteq X$, $|\overline{A}^\theta| \leqslant |A|^{\chi_\theta(X)}$.*

4. The almost-Lindelöf number

Definition 4.1. ([3]) The *almost-Lindelöf number* $\mathrm{aL}(A,X)$ of a set $A \subseteq X$ is the minimal cardinal κ such that, for each open cover $\mathcal{U}$ of A, there is $\mathcal{V} \in [\mathcal{U}]^{\leqslant\kappa}$ such that $A \subseteq \bigcup_{U\in\mathcal{U}} \overline{U}$.

Theorem 4.2. *Let X be a Hausdorff topological space. For each $A \subseteq X$,*

$$|A| \leqslant 2^{\mathrm{bts}_\theta(X)\cdot\chi_\theta(X)\cdot\mathrm{aL}(A,X)}.$$

Proof. Let $\kappa = \mathrm{bts}_\theta(X)\cdot\chi_\theta(X)\cdot\mathrm{aL}(A,X)$. For each $x \in X$, let $\mathcal{F}_x$ be a family of closed neighborhoods of x such that $|\mathcal{F}_x| \leqslant \kappa$, and each closed neighborhood of x contains one from $\mathcal{F}_x$.

Fix $a \in A$. We define, by induction on $\alpha \leqslant \kappa^+$, sets $A_\alpha \subseteq X$ such that $|A_\alpha| \leqslant 2^\kappa$.

$A_0 := \{a\}$.

Step $\alpha > 0$: Let $B = \bigcup_{\beta<\alpha} A_\beta$. By the induction hypothesis, $|B| \leqslant 2^\kappa$. Thus, $|\bigcup_{x\in B\cap A} \mathcal{F}_x| \leqslant 2^\kappa$ as well, and therefore $|[\bigcup_{x\in B\cap A} \mathcal{F}_x]^{\leqslant\kappa}| \leqslant 2^\kappa$. For each $\mathcal{V} \in [\bigcup_{x\in B\cap A} \mathcal{F}_x]^{\leqslant\kappa}$, with $A \setminus \bigcup\mathcal{V} \neq \emptyset$, pick a point from $A \setminus \bigcup\mathcal{V}$. Let C be the set of these points. Then $|B \cup C| \leqslant 2^\kappa$. Set $B_\alpha = \overline{B\cup C}^\theta$. As $\mathrm{bts}_\theta(X) \leqslant \kappa$, we have by Theorem 2.8 that $|B| \leqslant (2^\kappa)^\kappa = 2^\kappa$. End of the construction.

Let $B = B_{\kappa^+}$. It remains to show that $A \subseteq B$. Assume otherwise, and fix $a_0 \in A \setminus B$. As B is θ-closed, for each $x \in A \setminus B$ we can choose $V_x \in \mathcal{F}_x$ such that $V_x \cap B = \emptyset$. For $x \in A \cap B$, choose $V_x \in \mathcal{F}_x$ such that $a_0 \notin V_x$. As $\{V_x{}^\circ\colon x \in A\}$ is an open cover of A and $\mathrm{aL}(A,X) \leqslant \kappa$, there is $K \in [A]^{\leqslant\kappa}$ such that $A \subseteq \bigcup_{x\in K} V_x$. As $V_x \cap B = \emptyset$ for each $x \in A \setminus B$,

$$B \cap A \subseteq \bigcup_{x\in K\cap B} V_x.$$

As κ^+ is regular, there is $\alpha < \kappa^+$ such that $K\cap B \subseteq B_\alpha$. As $a_0 \in A\setminus\bigcup_{x\in K\cap B} V_x$, we have by the construction of $B_{\alpha+1}$ an element in $B_{\alpha+1} \cap A \setminus \bigcup_{x\in K\cap B} V_x$, and therefore so in $B \cap A \setminus \bigcup_{x\in K\cap B} V_x$; a contradiction. □

The following corollary improves upon a result of Bonanzinga, Cammaroto and Matveev [5], asserting that for Hausdorff, finitely-Urysohn spaces X, $|X| \leqslant 2^{\chi(X)\cdot\mathrm{aL}(X,X)}$.

Corollary 4.3. *Let X be a Hausdorff, finitely-Urysohn space. For each $A \subseteq X$, $|A| \leqslant 2^{\chi_\theta(X)\cdot\mathrm{aL}(A,X)}$. In particular, $|X| \leqslant 2^{\chi_\theta(X)\cdot\mathrm{aL}(X,X)}$.*

Proof. By Theorem 2.6, $\mathrm{bts}_\theta(X) \leqslant \mathrm{fbt}_\theta(X) \leqslant \chi_\theta(X)$ for finitely-Urysohn spaces. Thus, Theorem 4.2 applies. □

4.1. Final comment

Replacing, everywhere relevant, *closed neighborhoods* by *neighborhoods*, one obtains the notions of *finitely-Hausdorff* spaces, and the corresponding results hold true. This line of investigation was initiated by Bonanzinga in [4]. The results presented here generalize some of her results.

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