

Boaz Tsaban,*Department of Mathematics and Computer Science, Bar-Ilan University, 52900 Ramat-Gan, Israel. (e-mail: tsaban@macs.biu.ac.il)

A TOPOLOGICAL INTERPRETATION OF \mathfrak{t}

This paper is dedicated to Prof. Rabbi Haim Judah

Abstract

Hurewicz has found connections between some topological notions and the combinatorial cardinals \mathfrak{b} and \mathfrak{d} . Reclaw gave topological meaning to the definition of the cardinal \mathfrak{p} . We extend the picture with a topological interpretation of the cardinal \mathfrak{t} . We compare our notion to the one related to \mathfrak{p} , and to some other classical notions. This sheds new light on the famous open problem whether $\mathfrak{p} = \mathfrak{t}$.

1 Introduction

Cardinals associated with infinitary combinatorics play an important role in set theory. Some earlier works ([8], [13], [1], [12], and [11]) have pointed out a strong connection between these cardinals and classes of spaces having certain topological properties. In this paper, we continue this line of research in a way which enables us to give a topological meaning to an open problem from infinitary combinatorics.

1.1 Preliminaries

Let $\omega = \{0, 1, 2, \dots\}$ and $2 = \{0, 1\}$ be the usual discrete spaces. ω^ω and 2^ω are equipped with the product topology. Identify 2^ω with $P(\omega)$ via characteristic functions. $[\omega]^\omega$ is the set of infinite elements of $P(\omega)$, with $O_n = \{a : n \in a\}$ and $O_{\neg n} = \{a : n \notin a\} = O_n^c$ ($n \in \omega$) as a clopen subbase.

For $a, b \subseteq \omega$, $a \subseteq^* b$ if $a \setminus b$ is finite. $X \subseteq [\omega]^\omega$ is *centered* if every finite $F \subseteq X$ has an infinite intersection. $a \in [\omega]^\omega$ is an *almost-intersection* (a.i.) of X if it is infinite, and for all $b \in X$, $a \subseteq^* b$. $X \subseteq [\omega]^\omega$ is a *power* if it is

Key Words: \mathfrak{p} , \mathfrak{t} , γ -cover, small sets, λ -sets, infinitary combinatorics

Mathematical Reviews subject classification: 03E50, 03E10, 04A15

*This paper is based on my M.Sc. thesis work supervised by Martin Goldstern.

centered, but has no a.i.. \mathfrak{p} is the minimal size of a power. $X \subseteq [\omega]^\omega$ is a *tower* if it is linearly ordered by \subseteq^* , and has no a.i.¹ \mathfrak{t} is the minimal size of a tower.

\leq^* is the partial order defined on ω^ω by eventual dominance ($f \leq^* g$ iff $\forall^\infty n (f(n) \leq g(n))$). \mathfrak{b} is the minimal size of an unbounded family, and \mathfrak{d} is the minimal size of a dominating family, with respect to \leq^* .

The main problem

Let \mathfrak{c} denote the size of the continuum. The following holds.

Theorem 1.1 ([4, theorem 3.1.a]). $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

For each pair except \mathfrak{p} and \mathfrak{t} , it is well known that a strict inequality is consistent.

Problem 1.2. *Is $\mathfrak{p} < \mathfrak{t}$ consistent with ZFC ?*

This is one of the major and oldest problems of infinitary combinatorics. Allusions for this problem can be found in Rothberger's works (see, e.g., [15, lemma 7]). It is only known that $\mathfrak{p} = \omega_1 \rightarrow \mathfrak{t} = \omega_1$ [4, theorem 3.1.b], hence also $\mathfrak{t} \leq \omega_2 \rightarrow \mathfrak{p} = \mathfrak{t}$.

1.2 γ -spaces

Throughout this paper, by *space* we mean a zero-dimensional, separable, metrizable topological space. As any such space is homeomorphic to a subset of the irrationals, our results can be thought of as dealing with sets of reals.

The definition of a γ -space is due to Gerlits and Nagy [6]. Let X be a space. A collection of open sets \mathcal{G} is an ω -cover of X if for every finite $F \subseteq X$ there is a $G \in \mathcal{G}$ such that $F \subseteq G$. $\langle G_n : n < \omega \rangle$ is a γ -sequence for X if $\forall x \in X \forall^\infty n (x \in G_n)$. An open cover \mathcal{G} of X is a γ -cover of X if it contains a γ -sequence for X . Clearly every γ -cover is an ω -cover. X is a γ -space if every ω -cover of X is a γ -cover of X . For convenience, we may assume that the ω -covers \mathcal{G} of X are countable and clopen (replacing each element from \mathcal{G} by all finite unions of basic clopen sets contained in it), and that for all finite $F \subseteq X$, there are *infinitely many* $G \in \mathcal{G}$ with $F \subseteq G$ (using a partition of ω into ω many infinite sets, one can create a sequence consisting of ω copies of the original cover \mathcal{G} . The resulted sequence has the required property.) And so we will, from now on.

¹Note that, unlike the customary definition, we do not demand that a tower is well-ordered. However, by [4, theorem 3.7], this does not change the value of \mathfrak{t} .

Let Γ denote the collection of all γ -spaces. The γ -property is very strict: Gerlits and Nagy [6, corollary 6] proved that all γ -spaces are C'' . In particular, it is consistent that all γ -spaces are countable.

Reclaw has given an elegant characterization of γ -spaces.

Theorem 1.3 (Reclaw [13, theorem 3.2]). *X is a γ -space iff no continuous image of X in $[\omega]^\omega$ is a power.*

(A continuous image of X is the image of a continuous function with domain X .)

This gives an alternative proof to the following. For a family \mathcal{F} of spaces, let

$$\text{non}(\mathcal{F}) = \min\{|X| : X \text{ is a space, } X \notin \mathcal{F}\}$$

Corollary 1.4 (Galvin, Miller, Taylor [5, p. 146]). $\text{non}(\Gamma) = \mathfrak{p}$.

2 The tower of τ

Let X be a topological space. An open cover \mathcal{G} of X is T_1 if for all distinct $x, y \in X$ there is a $G \in \mathcal{G}$ such that $x \in G, y \notin G$. Therefore, \mathcal{G} is *not* T_1 iff there are distinct $x, y \in X$ such that for all $G \in \mathcal{G}$, $x \in G \rightarrow y \in G$. Strengthening the “not T_1 ” property demanding that the above holds for *all* $x, y \in X$ would trivialise \mathcal{G} to be $\{X\}$, or $\{X, \emptyset\}$. We therefore compensate by means of a “modulo finite” restriction.

For $\mathcal{G} = \langle G_n : n < \omega \rangle$, we write $x \overset{\mathcal{G}}{\rightsquigarrow} y$ for

$$\forall^\infty n (x \in G_n \rightarrow y \in G_n).$$

\mathcal{G} is a τ -sequence for X if

1. \mathcal{G} is a *large cover*, i.e. every element of X is covered by infinitely many elements of \mathcal{G} ,² and
2. for all $x, y \in X$, either $x \overset{\mathcal{G}}{\rightsquigarrow} y$, or $y \overset{\mathcal{G}}{\rightsquigarrow} x$.

An open cover \mathcal{J} of X is a τ -cover of X if it contains a τ -sequence for X . It is easy to see that every γ -cover is a τ -cover, and every τ -cover is an ω -cover.

X is a τ -space if every *clopen* τ -cover of X is a γ -cover of X . Equivalently, if every countable clopen τ -cover of X is a γ -cover of X . Let \mathcal{T} denote the collection of all τ -spaces.

²This requirement was added in order to avoid trivial cases.

Corollary 2.1. $\Gamma \subseteq \mathcal{T}$.

We wish to transfer our covering notions into $[\omega]^\omega$, in order to obtain their combinatorial versions. In Reclaw's proof of theorem 1.3, the following natural function $h = h_{\mathcal{G}}$ is considered: Given a countable large cover $\mathcal{G} = \{G_n : n \in \omega\}$ of X , define $h : X \rightarrow [\omega]^\omega$ by

$$h(x) = \{n : x \in G_n\}.$$

Now, let us see what h does to our topological notions. Assume that $\mathcal{G} = \langle G_n : n < \omega \rangle$ is an ω -cover of X . Then for all finite $F \subseteq X$, F is a subset of infinitely many G_n 's. That is, $n \in \cap h[F]$ for infinitely many n 's. This means that $h[X]$ is centered.

Next, assume that \mathcal{G} is a γ -sequence for X . Then $\forall x \forall^\infty n (x \in G_n)$. That is, $\forall x \forall^\infty n (n \in h(x))$, or: ω is an a.i. of $h[X]$. Therefore, \mathcal{G} is a γ -cover of X iff the associated $h[X]$ has an a.i..

Finally, a large cover \mathcal{G} is a τ -sequence for X iff for all $x, y \in X$, either $x \overset{\mathcal{G}}{\rightsquigarrow} y$, or $y \overset{\mathcal{G}}{\rightsquigarrow} x$. Now, $a \overset{\mathcal{G}}{\rightsquigarrow} b$ iff $\forall^\infty n (n \in h(a) \rightarrow n \in h(b))$ iff $h(a) \subseteq^* h(b)$. Therefore, $h[X]$ is linearly ordered by \subseteq^* .

We have showed the following.

Lemma 2.2. *Assume that \mathcal{G} is a countable large cover of X .*

1. \mathcal{G} is an ω -cover of X iff $h_{\mathcal{G}}[X]$ is centered.
2. \mathcal{G} is a γ -cover of X iff $h_{\mathcal{G}}[X]$ has an almost-intersection.
3. \mathcal{G} is a τ -sequence for X iff $h_{\mathcal{G}}[X]$ is linearly ordered by \subseteq^* .

Note that if \mathcal{G} is a clopen cover, then $h = h_{\mathcal{G}}$ is continuous, since for all n , $h^{-1}[O_n] = G_n$, and $h^{-1}[O_{-n}] = G_n^c$. Therefore, 2.2(1) and 2.2(2) yield Reclaw's theorem 1.3, and 2.2(2) and 2.2(3) yield the following.

Theorem 2.3. *X is a τ -space iff no continuous image of X in $[\omega]^\omega$ is a tower.*

Proof. (\Leftarrow) Assume that \mathcal{J} is a clopen τ -cover of X and let $\mathcal{G} \subseteq \mathcal{J}$ be a τ -sequence for X . Then by lemma 2.2(3), $h_{\mathcal{G}}[X]$ is linearly ordered by \subseteq^* . As $h_{\mathcal{G}}$ is continuous, $h_{\mathcal{G}}[X]$ cannot be a tower, and hence has an a.i.. Applying lemma 2.2(2), we get that \mathcal{G} is a γ -cover of X , and hence so is \mathcal{J} .

(\Rightarrow) Assume that X is a τ -space, and $f : X \rightarrow [\omega]^\omega$ is continuous. Assume that $f[X]$ is linearly ordered by \subseteq^* . Then $\langle O_n : n < \omega \rangle$ is a clopen τ -sequence for $f[X]$, therefore $\mathcal{G} = \langle f^{-1}[O_n] : n \in \omega \rangle$ is a clopen τ -sequence for X , hence a γ -cover of X . By 2.2(2) $h_{\mathcal{G}}[X]$ has an a.i., hence is not a tower. But $h_{\mathcal{G}} = f$, as for all $x \in X$,

$$n \in h_{\mathcal{G}}(x) \iff x \in f^{-1}[O_n] \iff f(x) \in O_n \iff n \in f(x).$$

Therefore, $f[X]$ is not a tower. \square

The reader might have noticed that in the last proof we have indirectly used the following lemma.

Lemma 2.4. *Every continuous image of a τ -space is a τ -space.*

Proof. A continuous preimage of a clopen τ -cover is a clopen τ -cover. \square

We get a topological characterization of \mathfrak{t} .

Corollary 2.5. $\text{non}(\mathcal{T}) = \mathfrak{t}$.

For a family \mathcal{F} of spaces, let

$$\text{add}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \& \ \bigcup \mathcal{A} \notin \mathcal{F}\}.$$

Theorem 2.6. $\text{add}(\mathcal{T}) = \mathfrak{t}$.

Proof. We need the following lemma.

Lemma 2.7. *Assume that $X \subseteq [\omega]^\omega$ is linearly ordered by \subseteq^* , and for some $\kappa < \mathfrak{t}$, $X = \bigcup_{i < \kappa} X_i$ where each X_i has an a.i.. Then X has an a.i..*

Proof. If for each $i < \kappa$ X_i has an a.i. $x_i \in X$, then an a.i. of $\{x_i : i < \kappa\}$ is also an a.i. of X .

Otherwise, there exists $i < \kappa$ such that X_i has no a.i. $x \in X$. That is, for all $x \in X$ there exists a $y \in X_i$ such that $x \not\subseteq^* y$, thus $y \subseteq^* x$. Therefore, an a.i. of X_i is also an a.i. of X . \square

Now we can use theorem 2.3. Assume that $X = \bigcup_{i < \kappa < \mathfrak{t}} X_i$, where each X_i is a τ -space. If $f : X \rightarrow [\omega]^\omega$ is continuous, and $f[X]$ is linearly ordered by \subseteq^* , then each $f[X_i]$ has an a.i. and therefore by the lemma, $f[X]$ has an a.i.. Therefore, X is a τ -space. \square

Unlike in the case of γ -spaces, large τ -spaces exist in *ZFC*. In fact, we have the following.

Theorem 2.8 (Shelah). 2^ω is a τ -space.

Proof. Towards a contradiction, assume that $f : 2^\omega \rightarrow [\omega]^\omega$ is continuous such that $f[2^\omega]$ is a tower.

Let

$$T = \{s \in 2^{<\omega} : f[[s]] \text{ has no a.i.}\}.$$

T is a perfect tree: Assume that for some $s \in T$, there are no incomparable extensions s_0, s_1 such that both $f[[s_0]]$ and $f[[s_1]]$ have no a.i.. Then for all \tilde{s} extending s , at least one of $f[[\tilde{s} \hat{\ } \langle 0 \rangle]]$ and $f[[\tilde{s} \hat{\ } \langle 1 \rangle]]$ has an a.i.. Let $\sigma \in 2^\omega$ extend s such that for all $n \geq |s|$, $f[[(\sigma \upharpoonright n) \hat{\ } \langle 1 - \sigma(n) \rangle]]$ has an a.i..

$f[[s]] = \bigcup_{n < \omega} f[[(\sigma \upharpoonright n)^\frown (1 - \sigma(n))]] \cup \{f(\sigma)\}$ is a union of ω many sets having an a.i., contradicting lemma 2.7.

We now show that $f[2^\omega]$ cannot be linearly ordered by \subseteq^* . Define two branches β and ξ in T as follows. Start with incomparable $b_0, c_0 \in T$. Pick $x_0 \in [b_0]$. As $f[[c_0]]$ has no a.i., we can find a $y_0 \in [c_0]$ such that $f(x_0) \not\subseteq^* f(y_0)$.

Choose an $n_0 \in f(x_0) \setminus f(y_0)$. Since f is continuous, we can find b_1 , an initial segment of x_0 , such that $f[[b_1]] \subseteq O_{n_0}$. Similarly, find c_1 , an initial segment of y_0 , such that $f[[c_1]] \subseteq O_{-n_0}$.

Now we reverse the roles, and find $x_1 \in [b_1]$, $y_1 \in [c_1]$, $n_1 > n_0$ such that $n_1 \in f(y_1) \setminus f(x_1)$. Then we take b_2 and c_2 , initial segments of y_1 and x_1 respectively, such that $f[[b_2]] \subseteq O_{-n_1}$ and $f[[c_2]] \subseteq O_{n_1}$.

We continue by induction.

Finally, let $\beta = \bigcup_i b_i = \lim_i x_i$, and $\xi = \bigcup_i c_i = \lim_i y_i$.

Since f is continuous, the sets $\{n_{2k} : k \in \omega\}$ and $\{n_{2k+1} : k \in \omega\}$ witness that neither $f(\beta) \subseteq^* f(\xi)$ nor $f(\xi) \subseteq^* f(\beta)$. \square

This theorem implies that the inclusion in corollary 2.1 is proper. We will modify it to get a large class of τ -spaces which are not γ -spaces.

Theorem 2.9. ω^ω is a τ -space.

Proof. Identify ω^ω with $2^\omega \setminus F$ (where F are the eventually zero sequences), and work in $2^\omega \setminus F$ instead of 2^ω :

1. In the proof that T is perfect, we need not care whether $\sigma \in 2^\omega \setminus F$ or not.
2. When choosing the initial segment b_{i+1} of x_i , use the fact that $x_i \notin F$ to make sure that b_{i+1} ends with a "1" (a similar treatment for c_{i+1}). This will make β and ξ belong to $2^\omega \setminus F$.

\square

Corollary 2.10. Every analytic set of reals is a τ -space.

Proof. Every analytic set of reals is a continuous image of ω^ω . \square

Remark 2.11.

1. One cannot prove in *ZFC* that all projective sets of reals are τ -spaces: Since the reals have a projective well-ordering in the constructible universe L , a straightforward inductive construction will yield a projective tower.

2. Due to a theorem of Suslin (see, e.g., [10, corollary 2C.3]), every uncountable analytic set contains a perfect set, and hence is not a γ -space (it is not even strongly null).

As in the case of γ -spaces [5, p. 147], the property of being a τ -space need not be hereditary for subspaces of the same size.

We will work in $P(\omega)$.

Theorem 2.12. $\mathfrak{t} = \mathfrak{c} \rightarrow$ *there is a space $X \subseteq [\omega]^\omega$ s.t.:*

1. $|X| = \mathfrak{c}$,
2. $X \cup [\omega]^{<\omega}$ is a τ -space, and
3. X is not a τ -space.

Proof. First, note that (1) follows from (2) and (3), using corollary 2.5.

We will use a modification of the Galvin-Miller construction (see [5, theorem 1]). For $y \in [\omega]^\omega$, define $y^* = \{x : x \subseteq^* y\}$. We need the following lemma.

Lemma 2.13 (Galvin, Miller [5, lemma 1.2]). *Assume that \mathcal{G} is an open ω -cover of $[\omega]^{<\omega}$. Then for all $x \in [\omega]^\omega$ there exists a $y \in [x]^\omega$ such that \mathcal{G} γ -covers y^* .*

Let $\langle \mathcal{G}_i : i < \mathfrak{c} \rangle$ enumerate all countable families of clopen sets in $P(\omega)$, and let $\langle y_i : i < \mathfrak{c} \rangle$ enumerate all elements $y \in [\omega]^\omega$ such that both y and $\omega \setminus y$ are infinite.

Construct, by induction, $\langle x_i : i < \mathfrak{c} \rangle \subseteq [\omega]^\omega$ such that $i < j \rightarrow x_j \subseteq^* x_i$. For a limit i , use $i < \mathfrak{t}$ to get x_i . For successor $i = k + 1$, x_i is constructed as follows:

Case 1 \mathcal{G}_k is a τ -cover of $B_k = \{x_j : j \leq k\} \cup [\omega]^{<\omega}$. By theorem 2.5, as $|B_k| < \mathfrak{t}$, \mathcal{G}_k is a γ -cover of B_k . In particular, \mathcal{G}_k γ -covers $[\omega]^{<\omega}$. By the lemma, there exists an $x_{k+1} \in [x_k]^\omega$ such that \mathcal{G}_k γ -covers x_{k+1}^* .

Case 2 \mathcal{G}_k is not a τ -cover of $\{x_j : j < k\} \cup [\omega]^{<\omega}$. Since this case is not interesting, we may take $x_{k+1} = x_k$.

After x_i is chosen (either for limit or successor i), modify it as follows: if $x_i \subseteq^* y_i$, leave it as is. Otherwise, replace it by $x_i \setminus y_i$.

This does the construction.

Define $X = \{x_i : i < \mathfrak{c}\}$, then $X \cup [\omega]^{<\omega}$ is a τ -space: by the construction, if \mathcal{G}_k is a τ -cover of $X \cup [\omega]^{<\omega}$, then it γ -covers $\{x_j : j \leq k\} \cup x_{k+1}^*$. But

$$X \cup [\omega]^{<\omega} \subseteq \{x_j : j \leq k\} \cup x_{k+1}^*.$$

This does (2).

(3) X is a tower: let $a \in [\omega]^\omega$. We will show that a is not an a.i. of X . Take $a_0 \subseteq a$ such that both a_0 and $\omega \setminus a_0$ are infinite. Now, some x_i satisfies either $x_i \subseteq^* a_0$, or $x_i \subseteq^* \omega \setminus a_0$. Therefore, $a \not\subseteq^* x_i$.

By lemma 2.4 (considering the identity function on $[\omega]^\omega$), X is not a τ -space. \square

Corollary 2.14. $\mathfrak{t} = \mathfrak{c} \rightarrow \tau$ -spaces are not closed under Borel images.

Proof. Let X be given by the theorem. Consider any function $f : X \cup [\omega]^{<\omega} \rightarrow [\omega]^\omega$ such that $f \upharpoonright X$ is the identity function, and $f[[\omega]^{<\omega}] \subseteq X$. As $[\omega]^{<\omega}$ is countable, f is Borel. $X \cup [\omega]^{<\omega}$ is a τ -space, but X , its Borel image, is not a τ -space. \square

3 Comparing τ -spaces to other classical classes

Hurewicz and Menger

We give Hurewicz' topological interpretations of \mathfrak{b} and \mathfrak{d} .

X has the *Hurewicz property* if for every sequence of open covers \mathcal{G}_n , there is a sequence of finite $\tilde{\mathcal{G}}_n \subseteq \mathcal{G}_n$ such that the sets $\cup \tilde{\mathcal{G}}_n$ form a γ -cover of X . X has the *Menger property* if for every sequence of open covers \mathcal{G}_n , there is a sequence of finite $\tilde{\mathcal{G}}_n \subseteq \mathcal{G}_n$ such that the sets $\cup \tilde{\mathcal{G}}_n$ cover X . Let \mathcal{H} and \mathcal{MEN} denote the classes of spaces having the Hurewicz and Menger properties, respectively.

Clearly $\mathcal{H} \subseteq \mathcal{MEN}$.

Theorem 3.1 (Hurewicz [8, §5]). *For every space X ,*

1. X has the Hurewicz property iff every continuous image of X in ω^ω is bounded. In particular, $\text{non}(\mathcal{H}) = \mathfrak{b}$.
2. X has the Menger property iff every continuous image of X in ω^ω is not dominating. In particular, $\text{non}(\mathcal{MEN}) = \mathfrak{d}$.

We get that none of these two notions is provably comparable to \mathcal{T} .

Corollary 3.2.

1. $\mathcal{T} \not\subseteq \mathcal{MEN}$, and
2. $\mathfrak{t} < \mathfrak{b} \rightarrow \mathcal{H} \not\subseteq \mathcal{T}$

Proof. (1) By 2.9, $\omega^\omega \in \mathcal{T}$, and by 3.1(2), $\omega^\omega \notin \mathcal{MEN}$.

(2) follows from 2.5 and 3.1(1). \square

Indeed, τ -spaces could be pretty far from having the Menger property. According to a theorem of Hurewicz [7, theorem 20], an analytic set of reals having the Menger property must be F_σ . Corollary 2.10 could be contrasted with this.

However, these classes need not be orthogonal. Gerlits and Nagy [6, p. 155] proved that, given a sequence of ω -covers \mathcal{G}_n of a γ -set X , there exists a sequence $G_n \in \mathcal{G}_n$ such that $\{G_n : n \in \omega\}$ γ -covers X . We therefore have:

Corollary 3.3. $\Gamma \subseteq \mathcal{H} \cap \mathcal{T}$.

λ -spaces

X is a λ -space if every countable subset of X is G_δ . Let Λ denote the collection of λ -spaces.

λ -spaces are perfectly meager (see [9, theorem 5.2]). Therefore, by remark 2.11(2), no uncountable analytic set is a λ -space. This again could be contrasted with corollary 2.10.

On the other hand, we have the following.

Theorem 3.4. *There is a λ -space of size \mathfrak{t} which is not a τ -space.*

Our theorem follows from the following two lemmas.

Lemma 3.5 ([16, theorem 1]). $\text{non}(\Lambda) = \mathfrak{b}$.

Lemma 3.6. *Every tower of size \mathfrak{b} is a λ -space.*

Proof. We use the standard argument (see [4, theorem 9.1]). Before getting started, note that for all $y \in [\omega]^\omega$, $y^* = \bigcup_{s \in [\omega]^{<\omega}} \{x : x \subseteq y \cup s\}$ is F_σ .

Assume that $X = \{x_i : i < \mathfrak{b}\}$ is a tower with $i < j \rightarrow x_j \subseteq^* x_i$. For $\alpha < \mathfrak{b}$, set $X_\alpha = \{x_i : i < \alpha\}$. Then each X_α is G_δ in X (its complement in X is F_σ).

Assume that $F \subseteq X$ is countable. As \mathfrak{b} is regular, there exists $\alpha < \mathfrak{b}$ such that $F \subseteq X_\alpha$. As $|X_\alpha| < \mathfrak{b}$, X_α is a λ -space. Hence, F is G_δ in X_α , i.e., there is a G_δ set $A \subseteq X$ such that $F = X_\alpha \cap A$. As X_α is also G_δ , F is G_δ in X . \square

With some set theoretic assumptions, we can have an example of size \mathfrak{b} . In fact, our \mathfrak{b} -example will have some additional properties related to our study. $X \subseteq \mathbb{R}$ is a λ' -space if for all countable $F \subseteq \mathbb{R}$, $X \cup F$ is a λ -space. For $D \subseteq \mathbb{R}$, X is κ -concentrated on D if for all open $U \supseteq D$, $|X \setminus U| < \kappa$.

Considering the proof that an (ω_1, ω_1) -gap is a λ' -space (see [9, p. 215]), one might wonder whether our proof can be strengthened to make every \mathfrak{b} -tower X a λ' -space. In fact, following the proof steps carefully one gets that for all countable $F \subseteq [\omega]^\omega$, $X \cup F$ is a λ -space. The problem is with $[\omega]^{<\omega}$:

if X , when viewed as a subset of ω^ω , is unbounded, then $[\omega]^{<\omega}$ is not G_δ in $X \cup [\omega]^{<\omega}$ [4, lemma 9.3].

Theorem 3.7. *Assume that there exists a tower of size \mathfrak{b} . Then there is a space X of size \mathfrak{b} such that:*

1. X is a λ -space,
2. X is \mathfrak{b} -concentrated on a countable set,
3. X is not a λ' -space,
4. X does not have the Hurewicz property, and
5. X is not a τ -space.

Proof. We work in $P(\omega)$. Identify $[\omega]^\omega$ with $\omega^{\uparrow\omega}$, the strictly increasing elements of ω^ω . Let $X = \{x_i : i < \mathfrak{b}\} \subseteq \omega^{\uparrow\omega}$ be such that:

(\clubsuit) It is unbounded, \leq^* -increasing, and has size \mathfrak{b} .

The existence of such a set follows from [4, theorem 3.3].

Let $A = \{a_i : i < \mathfrak{b}\}$ be a tower, and define $Y = \{y_i : i < \mathfrak{b}\}$ as follows: for each $i < \mathfrak{b}$, let $h \in \omega^\omega$ bound $\{y_k : k < i\} \cup \{x_i\}$, and take a $y_i \subseteq^* a_i$ such that $h \leq^* y_i$:

$$y_i(n) = \min\{k \in a_i : y_i(n-1), h(n) < k\}$$

Y , like X , has the property (\clubsuit). Rothberger [14, theorem 4] has proved that (\clubsuit) implies (1) and (3) (see also [4, lemma 9.3]). By an observation of Miller [9, theorem 5.7], (\clubsuit) implies that Y is \mathfrak{b} -concentrated on $[\omega]^{<\omega}$. By theorem 3.1(1), (4) is also satisfied.

(5) Y is a tower: any a.i. of Y would also be an a.i. of A . □

Our theorem has a cute corollary.

Corollary 3.8. $\mathfrak{t} = \mathfrak{b} \vee \mathfrak{b} < \mathfrak{d} \rightarrow$ *there exists an X as in theorem 3.7.*

This follows from the following observation.

Lemma 3.9 ([17, theorem 1]). $\mathfrak{b} < \mathfrak{d} \rightarrow$ *there is a tower of size \mathfrak{b} .*

For completeness, we give a proof of this lemma.

Proof. Let $X \subseteq \omega^{\uparrow\omega}$ have property (\clubsuit), and let $h \in \omega^\omega$ witness that X is not dominating. For each $x \in X$ define $a_x \in [\omega]^\omega$ by $a_x = \{n : x(n) < h(n)\}$. Then $\{a_x : x \in X\}$ is linearly ordered by \subseteq^* . Assume that it has an a.i. a . Then h' defined by $h'(n) = h(\min\{k \in a : n \leq k\})$ bounds X . A contradiction. □

Despite the large difference between τ and λ spaces, these classes need not be orthogonal. Their intersection could contain a space of size \mathfrak{c} .

Theorem 3.10 (Brendle [2, theorem 4.1]). *CH \rightarrow there is a γ -space of size $\mathfrak{c}(=\omega_1)$ all of whose subsets are γ -spaces.*

By [5, theorem 2], a G_δ γ -subspace of a space is also an F_σ subspace of that space. Every co-countable subspace of Brendle's space is G_δ and therefore F_σ . Therefore, every countable subspace of Brendle's space is G_δ .

Corollary 3.11. *CH \rightarrow there is a γ (in particular, τ)-space of size \mathfrak{c} which is also a λ -space.*

4 The selection principle S_1

Unlike γ -spaces, τ -spaces do not fit into the framework defined in [11]. We recall the basic definitions.

A space X has property $S_1(x, y)$ (x, y range over $\{\omega, \gamma, \tau, \dots\}$) if, given a sequence of x -covers \mathcal{G}_n , one can select from each \mathcal{G}_n an element G_n such that $\{G_n : n \in \omega\}$ is a y -cover. As mentioned in section 3, Gerlits and Nagy proved that the γ -property is equivalent to the $S_1(\omega, \gamma)$ property. Using this notation, we have the following.

Remark 4.1. $S_1(\omega, \gamma) \subseteq S_1(\tau, \gamma) \subseteq S_1(\gamma, \gamma)$.

Proof. As noted in §3, every γ -cover is a τ -cover, and every τ -cover is an ω -cover. \square

Obviously, $S_1(\tau, \gamma) \subseteq \mathcal{T}$. By [11, theorem 2.3], 2^ω does not belong to the class $S_1(\gamma, \gamma)$, and therefore not to the class $S_1(\tau, \gamma)$, either.

Corollary 4.2. $S_1(\tau, \gamma) \neq \mathcal{T}$.

We now study the $S_1(\tau, \gamma)$ property. Let us begin with saying that the τ -covering notion fits nicely into the framework of [11] (in fact, it suggests many interesting notions, but we will stick to $S_1(\tau, \gamma)$ in this paper). For example, it can be added to [11, theorem 3.1]. In particular, we have the following.

Theorem 4.3. $S_1(\tau, \gamma)$ is closed under taking closed subsets and continuous images.

There are more properties, which follow from remark 4.1 We quote some of them.

Theorem 4.4.

1. ([11, corollary 5.6]) *Every element of $S_1(\tau, \gamma)$ is perfectly meager (i.e., has meager intersection with every perfect set).*
2. ([11, theorem 5.7]) *If $X \in S_1(\tau, \gamma)$, then for every G_δ set G containing X , there exists an F_σ set F such that $X \subseteq F \subseteq G$.*

Remark 4.5.

1. If we omit the metrizability assumption on the spaces, then $S_1(\tau, \gamma)$ is not closed under cartesian products, nor under finite unions: Todorčević [18] showed that there exist (nonmetrizable) $X, Y \in S_1(\omega, \gamma)$ such that neither $X \times Y$ nor $X \cup Y$ belong to $S_1(\gamma, \omega)$.
2. Under CH , we can have such *metrizable* examples. This is achieved when applying the remarks of Galvin and Miller [5, p. 151] to Brendle's space (theorem 3.10).

Theorem 4.6 (Daniels [3, lemma 9]). $S_1(\omega, \gamma)$ is closed under taking finite powers.

Question 4.7. *Is $S_1(\tau, \gamma)$ closed under taking finite powers?*

One can see that if \mathcal{G} is a τ -cover of X , then $\{G^n : G \in \mathcal{G}\}$ is a τ -cover of X^n . But this is not enough for answering this question.

Theorem 4.8. $\text{non}(S_1(\tau, \gamma)) = \mathfrak{t}$.

Proof. Assume that $|X| < \mathfrak{t}$ and let \mathcal{G}_n be τ -covers of X . We wish to conclude that the \mathcal{G}_n 's are γ -covers of X . Corollary 2.5 is not enough for our purposes, since the τ -covers need not be clopen. However, theorem 2.2 gives the desired result.

Now, by [11, theorem 4.7], $\text{non}(S_1(\gamma, \gamma)) = \mathfrak{b}$. As $|X| < \mathfrak{b}$, $X \in S_1(\gamma, \gamma)$. Therefore, one can extract a γ -cover of X from the \mathcal{G}_n 's. This proves $\mathfrak{t} \leq \text{non}(S_1(\tau, \gamma))$.

The other direction follows from the fact that $S_1(\tau, \gamma) \subseteq \mathcal{T}$, together with corollary 2.5. \square

In particular, it is consistent that $S_1(\tau, \gamma) \neq S_1(\gamma, \gamma)$.

Corollary 4.9. $\mathfrak{t} < \mathfrak{b} \rightarrow S_1(\tau, \gamma) \neq S_1(\gamma, \gamma)$.

We therefore have the following.

Question 4.10. *Does $S_1(\omega, \gamma) = S_1(\tau, \gamma)$?*

As the consistency of $\mathfrak{p} < \mathfrak{t}$ would imply a negative answer, this question seems to be closely related to the main problem whether $\mathfrak{p} = \mathfrak{t}$.

Remark 4.11.

1. Due to 4.4(1), the (in fact, any) Luzin set used in [11] to distinguish $S_1(\omega, \omega)$ from $S_1(\omega, \gamma)$ will also distinguish it from $S_1(\tau, \gamma)$.
2. By 4.6, a negative answer to question 4.7 would imply a negative answer to question 4.10.
3. Showing $S_1(\tau, \gamma) \not\subseteq S_1(\omega, \omega)$ is consistent would also yield a negative answer.

Acknowledgements

I thank Martin Goldstern for supervising my thesis work (and the writing of this paper). I also thank Saharon Shelah for theorem 2.8, and Arnold W. Miller and Ireneusz Reclaw for showing interest in this work and making comments on it. A special thanks is owed to Marion Scheepers, for very fruitful discussions and suggestions which led to 2.12, 2.14, and §§3,4.

References

- [1] T. Bartszyński and H. Judah, *Borel images of sets of reals*, Real Analysis Exchange **20** (1994/5), pp. 536–558.
- [2] J. Brendle, *Generic constructions of small sets of reals*, Topology and its Applications **71** (1996), pp. 125–147.
- [3] P. Daniels, *Pixeley-Roy spaces over subsets of the reals*, Topology and its Applications **29** (1988), pp. 93–106.
- [4] E.K. van Douwen, *The integers and topology*, in: **Handbook of Set Theoretic Topology** (K. Kunen and J. Vaughan, Eds.), North-Holland, Amsterdam, 1984, pp. 111–167.
- [5] F. Galvin and A.W. Miller, *γ -sets and other singular sets of real numbers*, Topology and its Applications **17** (1984), pp. 145–155.
- [6] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I*, Topology and its applications **14** (1982), pp. 151–161.
- [7] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Mathematische Zeitschrift **24** (1925), pp. 401–421.
- [8] —, *Über Folgen stetiger Funktionen*, Fundamenta Mathematicae **9** (1927), pp. 193–204.

- [9] A.W. Miller, *Special subsets of the real line*, in: **Handbook of Set Theoretic Topology** (K. Kunen and J. Vaughan, Eds.), North-Holland, Amsterdam, 1984, pp. 201–233.
- [10] Y.N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.
- [11] W. Just, A.W. Miller, M. Scheepers, and P. J. Szeptycki, *The combinatorics of open covers II*, *Topology and its applications* **73** (1996), pp. 241–266.
- [12] J. Pawlikowski and I. Reclaw, *Parametrized Cichon's diagram and small sets*, *Fundamenta Mathematicae* **147** (1995), pp. 135–155.
- [13] I. Reclaw, *Every Luzin set is undetermined in point-open game*, *Fundamenta Mathematicae* **144** (1994), pp. 43–54.
- [14] F. Rothberger, *Sur un ensemble toujours de première catégorie qui est dépourvu de la propriété λ* , *Fundamenta Mathematicae* **32** (1939), pp. 294–300.
- [15] —, *On some problems of Hausdorff and of Seierpiński*, *Fundamenta Mathematicae* **35** (1948), pp. 29–46.
- [16] —, *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C* , *Proceedings of the Cambridge Philosophical Society* **37** (1941), pp. 109–126.
- [17] Z. Shuguo, *Relations among cardinal invariants*, in: *Proceedings of Chinese Conference on Pure and Applied Logic* (ed. Z. Jinwen), Beijing 1992, pp. 145–147.
- [18] S. Todorcević, *Aronszajn Orderings*, *Publications de L'Institut Mathématique* **57** (1995), pp. 29–46.