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ON THE PYTKEEV PROPERTY IN SPACES OF CONTINUOUS FUNCTIONS (II)

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Abstract. We prove that for each Polish space X, the space $C(X)$ of continuous real-valued functions on X satisfies (a strong version of) the Pytkeev property, if endowed with the compact-open topology. We also consider the Pytkeev property in the case where $C(X)$ is endowed with the topology of pointwise convergence.

1. Introduction

For a topological space X, $C(X)$ is the family of all real-valued continuous functions on X. We consider two standard topologies on $C(X)$, which make it a topological group. Let $\mathbf{0}$ denote the constant zero function on X.

$C_k(X)$ denotes $C(X)$, endowed with the compact-open topology. For a set $K \subseteq X$ and $n \in \mathbb{N}$, let

$$[K;n] = \left\{ f \in C_k(X) : (\forall x \in K)\ |f(x)| < \frac{1}{n} \right\}.$$

When K ranges over the compact subsets of X and n ranges over $\mathbb{N}$, the sets $[K;n]$ form a local base at $\mathbf{0}$.

$C_p(X)$ denotes $C(X)$, endowed with the topology of pointwise convergence. Here, a local base at $\mathbf{0}$ is given by the sets $[F;n]$, where $n \in \mathbb{N}$, and F ranges over the finite subsets of X.

$C_k(X)$ is metrizable if, and only if, X is hemicompact (i.e., there is a countable family of compact sets such that each compact subset of X is contained in some

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member of the family) [9]. In particular, $C_k(\mathbb{N}^\mathbb{N})$ is not metrizable. Restricting attention to first countable spaces X, McCoy [9] observed that for $C_k(X)$ to be metrizable, it suffices that it has the *Fréchet-Urysohn* property, that is, for each $A \subseteq C_k(X)$ with $\mathbf{0} \in \overline{A}$, there is a sequence of elements of A converging to $\mathbf{0}$. Despite the fact that $C_k(\mathbb{N}^\mathbb{N})$ does not have the Fréchet-Urysohn property, we show in Section 2 that it has the slightly weaker Pytkeev property.

As for $C_p(X)$, it is metrizable if, and only if, X is countable [1]. Here, the Fréchet-Urysohn property does not imply metrizability, and Sakai asked whether for $C_p(X)$, the Pytkeev property implies the Fréchet-Urysohn property. We establish several weaker assertions (Section 3).

2. The compact-open topology

Let X be a topological space. $C_k(X)$ has the *Pytkeev property* [11] if for each $A \subseteq C_k(X)$ with $\mathbf{0} \in \overline{A} \setminus A$, there are infinite sets $A_1, A_2, \ldots \subseteq A$ such that each neighborhood of $\mathbf{0}$ contains some A_n.

The notion of a *k-cover* is central in the study of local properties of $C_k(X)$ (see [3] and references therein). A cover $\mathcal{U}$ of X is a k-cover of X if $X \notin \mathcal{U}$, but for each compact $K \subseteq X$, there is $U \in \mathcal{U}$ such that $K \subseteq U$.

Theorem 2.1. *$C_k(\mathbb{N}^\mathbb{N})$ has the Pytkeev property.*

Proof. By a theorem of Pavlovic and Pansera [10], it suffices to prove that for each open k-cover $\mathcal{U}$ of X, there are infinite sets $\mathcal{U}_1, \mathcal{U}_2, \ldots \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is a k-cover of X. We will show that $\mathbb{N}^\mathbb{N}$ has the mentioned covering property.

To this end, we set up some basic notation. For $s \in \mathbb{N}^{<\aleph_0}$, $[s] = \{f \in \mathbb{N}^\mathbb{N} : s \subseteq f\}$, and $|s|$ denotes the length of s. For $S \subseteq \mathbb{N}^{<\aleph_0}$, $[S] = \bigcup_{s\in S}[s]$. For an open $U \subseteq \mathbb{N}^\mathbb{N}$, $U(n) = \{s \in \mathbb{N}^n : [s] \subseteq U\}$. Note that for each n, $[U(n)] \subseteq [U(n+1)]$, and $U = \bigcup_n [U(n)]$.

Lemma 2.2. *Assume that $\mathcal{U}$ is an open k-cover of $\mathbb{N}^\mathbb{N}$. Then:*

1. *$\mathcal{V} = \{[U(n)] : U \in \mathcal{U}, n \in \mathbb{N}\}$ is a k-cover of $\mathbb{N}^\mathbb{N}$.*
2. *There is n such that $\{U(n) : U \in \mathcal{U}\}$ is infinite.*
3. *For each compact $K \subseteq \mathbb{N}^\mathbb{N}$, there is n such that $\{U(n) : U \in \mathcal{U}, K \subseteq [U(n)]\}$ is infinite.*

Proof. (1) For each compact $K \subseteq \mathbb{N}^\mathbb{N}$, there is $U \in \mathcal{U}$ such that $K \subseteq U$. As $U = \bigcup_n [U(n)]$ and K is compact, there is n such that $K \subseteq [U(n)] \in \mathcal{V}$.

(2) Assume that for each n, $\{U(n) : U \in \mathcal{U}\}$ is finite. Note that for each $U \in \mathcal{U}$ and each n, $[U(n)] \subseteq U \neq \mathbb{N}^{\mathbb{N}}$, and therefore $U(n) \neq \mathbb{N}^n$. Proceed by induction on n:

Step 1. As $\mathcal{U}(1) = \{U(1) : U \in \mathcal{U}\}$ is finite and $\mathbb{N} \notin \mathcal{U}(1)$, there is a finite $F_1 \subseteq \mathbb{N}$ which is not contained in any member of $\mathcal{U}(1)$.

Step n. As $\mathcal{U}(n) = \{U(n) : U \in \mathcal{U}\}$ is finite and $F_{n-1} \times \mathbb{N}$ is not contained in any member of $\mathcal{U}(n)$, there is a finite $F_n \subseteq F_{n-1} \times \mathbb{N}$ which is not contained in any member of $\mathcal{U}(n)$, and such that $F_n \restriction (n-1) = F_{n-1}$.

Take $K = \bigcap_n [F_n]$ (the set of all infinite branches through the finitely splitting tree $\bigcup_n F_n$). As K is compact, there is $U \in \mathcal{U}$ such that $K \subseteq U$. As $U = \bigcup_n [U(n)]$ and K is compact, there is n such that $K \subseteq [U(n)]$. But then $F_n \subseteq U(n)$, a contradiction.

(3) By (1), $\{[U(n)] : U \in \mathcal{U}, n \in \mathbb{N}, K \subseteq [U(n)]\}$ is a k-cover of $\mathbb{N}^{\mathbb{N}}$. By (2), there is m such that

$$\mathcal{V} = \{[[U(n)](m)] : U \in \mathcal{U}, n \in \mathbb{N}, K \subseteq [U(n)]\}$$

is infinite. For all U and n, $[[U(n)](m)]$ is equal to $[U(n)]$ when $n \leq m$, and to $[U(m)]$ when $m < n$. Thus, $\mathcal{V} = \bigcup_{n \leq m} \{[U(n)] : U \in \mathcal{U}, K \subseteq [U(n)]\}$, and therefore there is $n \leq m$ such that $\{[U(n)] : U \in \mathcal{U}, K \subseteq [U(n)]\}$ is infinite. □

For each n and $s \in \mathbb{N}^n$, let $[\leq s] = [\{t \in \mathbb{N}^n : t \leq s\}]$, where $\leq$ is pointwise. The following lemma gives more than what is needed in our theorem.

Lemma 2.3. *Let $\mathcal{U}$ be an open k-cover of $\mathbb{N}^{\mathbb{N}}$. There is $S \subseteq \mathbb{N}^{<\aleph_0}$ such that for each $s \in S$, $\mathcal{U}_s = \{U \in \mathcal{U} : [\leq s] \subseteq U\}$ is infinite, and $\{[\leq s] : s \in S\}$ is a clopen k-cover of $\mathbb{N}^{\mathbb{N}}$ (refining $\{\bigcap \mathcal{U}_s : s \in S\}$).*

Proof. We actually prove the stronger result, that the statement in the lemma holds when

$$\mathcal{U}_s = \{[U(|s|)] : U \in \mathcal{U}, [\leq s] \subseteq U\}$$

for each $s \in S$.

Let S be the set of all $s \in \mathbb{N}^{<\aleph_0}$ such that $\mathcal{U}_s$ is infinite. If $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact, take $f \in \mathbb{N}^{\mathbb{N}}$ such that the compact set $K(f) = \{g \in \mathbb{N}^{\mathbb{N}} : g \leq f\}$ contains K. By Lemma 2.2, there is n such that there are infinitely many sets $U(n)$, $U \in \mathcal{U}$, with $K(f) \subseteq [U(n)]$, that is, $[\leq f \restriction n] \subseteq U$. Thus, $f \restriction n \in S$. Clearly, $K \subseteq K(f) \subseteq [\leq f \restriction n]$. □

This completes the proof of Theorem 2.1. □

Definition 1. For shortness, we say that a topological space X is *nice* if there is a countable family $\mathcal{C}$ of open subsets of X, such that for each open k-cover $\mathcal{U}$ of X, $\mathcal{S} = \{V \in \mathcal{C} : (\exists^\infty U \in \mathcal{U})\ V \subseteq U\}$ is a k-cover of X.

By Lemma 2.3, $\mathbb{N}^\mathbb{N}$ is nice.

Definition 2. A topological space Y has the *strong Pytkeev property* if for each $y \in Y$, there is a *countable* family $\mathcal{N}$ of subsets of Y, such that for each neighborhood U of y and each $A \subseteq Y$ with $y \in \overline{A} \setminus A$, there is $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite.

If Y is first countable, then it has the strong Pytkeev property. The converse fails, even in the realm of $C_k(X)$. Indeed, $C_k(\mathbb{N}^\mathbb{N})$ is not first countable (since it is a non-metrizable topological group), and we have the following.

Theorem 2.4. *$C_k(\mathbb{N}^\mathbb{N})$ has the strong Pytkeev property.*

Theorem 2.4 follows from the following.

Lemma 2.5. *If X is nice, then $C_k(X)$ has the strong Pytkeev property.*

PROOF. Let $\mathcal{C}$ be as in the definition of niceness for X. It suffices to verify the strong Pytkeev property of $C_k(X)$ at $\mathbf{0}$. Set

$$\mathcal{N} = \{[V; n] : V \in \mathcal{C}, n \in \mathbb{N}\}.$$

Assume that $A \subseteq C_k(X)$ and $\mathbf{0} \in \overline{A} \setminus A$. There are two cases to consider.

Case 1: For each n, there is $f_n \in A \cap [X; n]$ (equivalently, there are infinitely many such n). Given any neighborhood $[K; m]$ of $\mathbf{0}$, take $V \in \mathcal{C}$ with $K \subseteq V$. Then $[V; m] \subseteq [K; m]$, and $[V; m] \cap A \supseteq \{f_n : n \geq m\}$ is infinite.

Case 2: There is N such that for each $n \geq N$, $A \cap [X; n] = \emptyset$. Fix $n \geq N$. $\mathcal{U}_n = \{f^{-1}[(-1/n, 1/n)] : f \in A\}$ is a k-cover of X. Thus,

$$\begin{aligned} \mathcal{S}_n &= \{V \in \mathcal{C} : (\exists^\infty U \in \mathcal{U}_n)\ V \subseteq U\} \subseteq \\ &\subseteq \{V \in \mathcal{C} : (\exists^\infty f \in A)\ V \subseteq f^{-1}[(-1/n, 1/n)]\} = \\ &= \{V \in \mathcal{C} : [V; n] \cap A \text{ is infinite}\} \end{aligned}$$

is a k-cover of X.

Consider any (basic) open neighborhood $[K; n]$ of $\mathbf{0}$. Take $V \in \mathcal{S}_n$ such that $K \subseteq V$. Then $[V; n] \in \mathcal{N}$, $[V; n] \subseteq [K; n]$, and $[V; n] \cap A$ is infinite. □

A function $f : X \to Y$ is *compact-covering* if for each compact $K \subseteq Y$, there is a compact $C \subseteq X$ such that $K \subseteq f[C]$. Hereditary local properties of a space $C_k(X)$ are clearly preserved when transforming X by a continuous compact-covering functions. (Indeed, if $f : X \to Y$ is a continuous compact-covering surjection, then $g \mapsto g \circ f$ is an embedding of $C_k(Y)$ into $C_k(X)$.)

Corollary 2.6. *For each Polish space X, $C_k(X)$ has the strong Pytkeev property.*

PROOF. X is the image of $\mathbb{N}^{\mathbb{N}}$ under a continuous compact-covering function. Indeed [7]: There is a closed $C \subseteq \mathbb{N}^{\mathbb{N}}$ such that X is the image of C under a perfect (thus compact-covering) function. As C is closed, it is a retract of $\mathbb{N}^{\mathbb{N}}$, and the retraction is clearly compact covering. $\square$

3. The topology of pointwise convergence

There is a very rich local-to-global theory, due to Arhangel'skiĭ and his followers, which studies local properties of $C_p(X)$ by translating them into covering properties. An elegant and uniform treatment of covering properties was given by Scheepers [16, 6]. We recall a part of this theory that puts the results of the present section in their proper context.

Let X be a topological space. $\mathcal{U}$ is a *cover* of X if $X = \bigcup \mathcal{U}$ but $X \notin \mathcal{U}$. A cover $\mathcal{U}$ of X is an *ω-cover* of X if for each finite subset F of X, there is $U \in \mathcal{U}$ such that $F \subseteq U$. $\mathcal{U}$ is a *γ-cover* of X if it is infinite and for each x in X, $x \in U$ for all but finitely many $U \in \mathcal{U}$. Let $\mathcal{O}$, Ω, and Γ denote the collections of all open covers, ω-covers, and γ-covers of X, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space X. Following are selection hypotheses which X may satisfy or not satisfy [16].

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathcal{A}$, there are $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots$, such that $\{U_1, U_2, \dots\} \in \mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathcal{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$, such that $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathcal{B}$.

$\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathcal{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$, such that $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots\} \in \mathcal{B}$.

Some of the properties defined in this manner were studied earlier by Hurewicz ($\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$), Menger ($\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$), Rothberger ($\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, traditionally known as the C'' property), Gerlits and Nagy ($\mathsf{S}_1(\Omega, \Gamma)$, traditionally known as the γ-property), and others. Each of these properties is either trivial, or equivalent to one in Figure 1 (where an arrow denotes implication) [6].

In the remainder of this paper, all spaces X are assumed to be Tychonoff. A space X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ if, and only if, $C_p(X)$ has the Fréchet-Urysohn property [5]. In particular, if X satisfies $\mathsf{S}_1(\Omega, \Gamma)$, then $C_p(X)$ has the Pytkeev property.

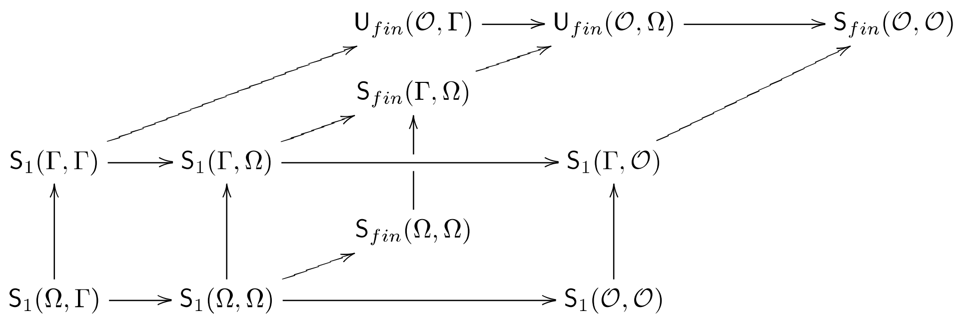

FIGURE 1. The Scheepers Diagram

Problem 3.1 (Sakai [14]). *Assume that $C_p(X)$ has the Pytkeev property. Must X satisfy $\mathsf{S}_1(\Omega,\Gamma)$?*

For metric spaces X which are countable unions of totally bounded subspaces, Miller proved that consistently, X is countable whenever $C_p(X)$ has the Pytkeev property (this is essentially proved in Theorem 18 of [18]). It follows that a positive answer to Sakai's Problem 3.1 is consistent in this realm. However, we suspect that the following holds.

Conjecture 3.2 (CH). *There is $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that $C_p(X)$ has the Pytkeev property, but X does not even satisfy Menger's property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.*

It is therefore natural to consider the conjunction of "$C_p(X)$ has the Pytkeev property" with properties in the Scheepers Diagram 1.

A combination of results of Kočinac and Scheepers [8] and Sakai [14] gives that if $C_p(X)$ has the Pytkeev property and X satisfies $\mathsf{S}_{fin}(\Omega,\Omega)$, then all finite powers of X satisfy $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ as well as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. We will prove several results of a similar flavor.

The combinatorial terminology in the remainder of the paper is as follows: For $f,g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n. $B \subseteq \mathbb{N}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that for each $f \in B$, $f \leq^* g$. $D \subseteq \mathbb{N}^{\mathbb{N}}$ is *finitely dominating* if its closure under pointwise maxima of finite subsets is dominating.

Theorem 3.3. *If $C_p(X)$ has the Pytkeev property and X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$, then X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ as well as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

PROOF. As $C_p(X)$ has the Pytkeev property, X is Lindelöf and zero-dimensional [13]. This is needed for the application of the quoted combinatorial theorems below.

We first prove that X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$. By [12], it suffices to prove the following.

Lemma 3.4. *If $C_p(X)$ has the Pytkeev property and X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$, then each continuous image Y of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. Let Y be a continuous image of X in $\mathbb{N}^{\mathbb{N}}$. Since we can transform Y continuously by $f(n) \mapsto f(0) + f(1) + \cdots + f(n) + n$, we may assume that all elements of Y are increasing. If there is an infinite $I \subseteq \mathbb{N}$ such that $\{f \restriction I : f \in Y\}$ is bounded, then Y is bounded. We therefore assume that there is N such that for each $n \geq N$, $\{f(n) : f \in Y\}$ is infinite.

As Y satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$, Y is not finitely dominating [19], that is, there is $g \in \mathbb{N}^{\mathbb{N}}$ such that the clopen sets $U_n = \{f \in Y : f(n) \leq g(n)\}$, $n \geq N$, form an ω-cover of Y. As $C_p(Y)$ has the Pytkeev property, there are infinite $I_1, I_2, \ldots \subseteq \mathbb{N} \setminus \{0,\ldots,N-1\}$ such that $\{\bigcap_{k\in I_n} U_k : n \in \mathbb{N}\}$ is an ω-cover of Y [13]. For each n, $\{f \restriction I_n : f \in \bigcap_{k\in I_n} U_k\}$ is bounded, and therefore $\bigcap_{k\in I_n} U_k$ is bounded. Thus, $Y = \bigcup_n \bigcap_{k\in I_n} U_k$ is bounded. □

We now show that X satisfies $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. It suffices to prove that each continuous image Y of X in $\mathbb{N}^{\mathbb{N}}$ has strong measure zero with respect to the standard metric of $\mathbb{N}^{\mathbb{N}}$ [4]. Indeed, by Lemma 3.4, such an image Y is bounded, and thus is a countable union of totally bounded subspaces of $\mathbb{N}^{\mathbb{N}}$. By a theorem of Miller [18], if $C_p(Y)$ has the Pytkeev property and Y is a countable union of totally bounded subspaces, then Y has strong measure zero. □

$\mathfrak{D}_{\text{fin}}$ is the family of all subsets of $\mathbb{N}^{\mathbb{N}}$ which are not finitely dominating, and $\text{cov}(\mathfrak{D}_{\text{fin}}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathfrak{D}_{\text{fin}} \text{ and } \bigcup\mathcal{F} = \mathbb{N}^{\mathbb{N}}\}$. The hypothesis $\text{cov}(\mathfrak{D}_{\text{fin}}) < \mathfrak{d}$ holds, e.g., in the Cohen reals model, or if $\mathfrak{d}$ is singular [17].

Theorem 3.5 ($\text{cov}(\mathfrak{D}_{\text{fin}}) < \mathfrak{d}$). *Assume that for each $Y \subseteq X$, $C_p(Y)$ has the Pytkeev property. Then X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ as well as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

Proof. By Theorem 3.3, it suffices to prove that X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$, or equivalently, that no continuous image Y of X in $\mathbb{N}^{\mathbb{N}}$ is finitely dominating.

Assume that Y is a continuous image of X in $\mathbb{N}^{\mathbb{N}}$. We may assume that all elements of Y are increasing. Let $\kappa = \text{cov}(\mathfrak{D}_{\text{fin}}) < \mathfrak{d}$, and $Y_\alpha \subseteq \mathbb{N}^{\mathbb{N}}$, $\alpha < \kappa$, be not finitely dominating and such that $\bigcup_{\alpha<\kappa} Y_\alpha = \mathbb{N}^{\mathbb{N}}$. For each $\alpha < \kappa$, $Y \cap Y_\alpha$ is not finitely dominating, and since it is a continuous image of a subset of X, $C_p(Y \cap Y_\alpha)$ has the Pytkeev property. The proof of Lemma 3.4 shows the following.

Lemma 3.6. *Assume that $Z \subseteq \mathbb{N}^{\mathbb{N}}$, all elements of Z are increasing, Z is not finitely dominating, and $C_p(Z)$ has the Pytkeev property. Then Z is bounded.* □

It follows that $Y \cap Y_\alpha$ is bounded for all $\alpha < \kappa$, and as $\kappa < \mathfrak{d}$, $Y = \bigcup_{\alpha<\kappa} Y \cap Y_\alpha$ is not finitely dominating. □

We now consider the *strong* Pytkeev property of $C_p(X)$. A space Y has a *countable cs*-character* [2] if for each $y \in Y$, there is a countable family $\mathcal{N}$ of subsets of Y, such that for each sequence in Y converging to y (but not eventually equal to y) and each neighborhood U of y, there is $N \in \mathcal{N}$ such that $N \subseteq U$ and N contains infinitely many elements of that sequence. Clearly, the strong Pytkeev property implies countable cs*-character. For topological groups, the conjunction of countable cs*-character and the Fréchet-Urysohn property implies metrizability [2]. As $C_p(X)$ is a topological group, we have the following.

Corollary 3.7. *If $C_p(X)$ has the Fréchet-Urysohn property as well as the strong Pytkeev property, then X is countable.* □

As the Pytkeev property follows from the Fréchet-Urysohn property, we have the following.

Corollary 3.8. *The Pytkeev property for $C_p(X)$ does not imply the strong Pytkeev property for $C_p(X)$.* □

If, consistently, there is an uncountable X such that $C_p(X)$ has the strong Pytkeev property, then the answer to Sakai's Problem 3.1 is negative: By corollary 3.7, in this case $C_p(X)$ cannot have the Fréchet-Urysohn property.[1]

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