

## The Hurewicz covering property and slaloms in the Baire space

by

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**Abstract.** According to a result of Kočinac and Scheepers, the Hurewicz covering property is equivalent to a somewhat simpler selection property: For each sequence of large open covers of the space one can choose finitely many elements from each cover to obtain a groupable cover of the space. We simplify the characterization further by omitting the need to consider sequences of covers: A set of reals  $X$  has the Hurewicz property if, and only if, each large open cover of  $X$  contains a groupable subcover. This solves in the affirmative a problem of Scheepers. The proof uses a rigorously justified abuse of notation and a “structure” counterpart of a combinatorial characterization, in terms of slaloms, of the minimal cardinality  $\mathfrak{b}$  of an unbounded family of functions in the Baire space. In particular, we obtain a new characterization of  $\mathfrak{b}$ .

**1. Introduction.** A separable zero-dimensional metrizable space  $X$  has the *Hurewicz property* [3] if:

For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of open covers of  $X$  there exist finite subsets  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that  $X \subseteq \bigcup_n \bigcap_{m > n} \bigcup \mathcal{F}_m$ .

This property is a generalization of  $\sigma$ -compactness.

Much effort was put in the past in order to find a simpler characterization of this property. In particular, it was desired to avoid the need to glue the elements of each  $\mathcal{F}_n$  together (that is, by taking their union) in the definition of the Hurewicz property.

The first step toward simplification was the observation that one may restrict attention to sequences of *large* (rather than arbitrary) open covers of  $X$  in the above definition [10] ( $\mathcal{U}$  is a *large cover* of  $X$  if each member of  $\mathcal{U}$  is contained in infinitely many members of  $\mathcal{U}$ ).

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The main ingredient in the next major step toward this goal was implicitly studied in [11, 8, 5] while considering spaces having analogous properties in all finite powers and a close relative of the Reznichenko (or: weak Fréchet–Urysohn) property. Finally, this ingredient was isolated and analyzed in [6]: A large cover  $\mathcal{U}$  of  $X$  is *groupable* if there exists a partition  $\mathcal{P}$  of  $\mathcal{U}$  into finite sets such that for each  $x \in X$  and all but finitely many  $\mathcal{F} \in \mathcal{P}$ ,  $x \in \cup \mathcal{F}$ . Observe that ignoring all but countably many elements of the partition, we see that each groupable cover contains a countable groupable cover. Moreover, in [13] it is proved that for the types of spaces considered here, each large open cover contains a countable large cover.

One of the main results in Kočinac–Scheepers’s [6] is that the Hurewicz property is equivalent to the following one:

For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of large open covers of  $X$  there exist finite subsets  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that  $\bigcup_n \mathcal{F}_n$  is a groupable cover of  $X$ .

This characterization is misleading in its pretending to be a mere result of unstitching the unions  $\cup \mathcal{F}_n$ : The sets  $\mathcal{F}_n$  need not be disjoint in the original definition, and overcoming this difficulty requires a deep analysis involving infinite game-theoretic methods—see [6].

In this paper we take the task of simplification one step further by removing the need to consider *sequences* of covers. We prove that the Hurewicz property is equivalent to:

( $\star$ ) Each large cover of  $X$  contains a groupable subcover.

This solves in the affirmative a problem of Scheepers [11, Problem 1], which asks whether, for strong measure zero sets, ( $\star$ ) is equivalent to the Hurewicz property.

Another way to view this simplification is as follows: The Kočinac–Scheepers characterization is equivalent to requiring that the resulting cover  $\bigcup_n \mathcal{F}_n$  is large *together* with the property that each large open cover of  $X$  contains a groupable cover of  $X$ . The first requirement has also appeared in the literature, and is equivalent to a property introduced by Menger in [7] (see [10]). Our result says that it is enough to require only that the second property holds, or in other words, that the second property actually implies the first.

**2. Two possible interpretations.** Our proof relies on a delicate interplay between two possible interpretations of the term “large open cover of  $X$ ” when  $X$  is a subspace of another space  $Y$ :

- (1) A large open cover of  $X$  by subsets of  $X$  which are relatively open in  $X$ ; and

(2) A large open cover of  $X$  by open subsets of  $Y$ .

The notions do not coincide, because a large cover of the second type, when restricted to  $X$ , need not be large—it can even be finite.

For brevity, we will use the following notation. For a space  $X$ , denote the property that each large cover of  $X$  by open subsets of  $X$  contains a groupable cover of  $X$  by  $(\overset{A}{\underset{A^{\text{gp}}}{X}})$ . We write  $(\overset{A}{\underset{A^{\text{gp}}}{X}})$  instead of  $(\overset{A_X}{\underset{A^{\text{gp}}_X}{X}})$  when the space  $X$  is clear from the context. It is easy to see that  $(\overset{A_X}{\underset{A^{\text{gp}}_X}{X}})$  implies that each *countable* large open cover of  $X$  is groupable (divide the countably many remaining elements between the sets in the partition so that they remain finite).

We will need the following simple fact.

LEMMA 1. *The property  $(\overset{A}{\underset{A^{\text{gp}}}{X}})$  is preserved under taking closed subsets and continuous images, that is:*

- (1) *If  $(\overset{A_X}{\underset{A^{\text{gp}}_X}{X}})$  holds and  $C$  is a closed subset of  $X$ , then  $(\overset{A_C}{\underset{A^{\text{gp}}_C}{C}})$  holds.*
- (2) *If  $(\overset{A_X}{\underset{A^{\text{gp}}_X}{X}})$  holds and  $Y$  is a continuous image of  $X$ , then  $(\overset{A_Y}{\underset{A^{\text{gp}}_Y}{Y}})$  holds.*

*Proof.* (1) Assume that  $\mathcal{U}$  is a large open cover of  $C$ . Then  $\tilde{\mathcal{U}} = \{U \cup (X \setminus C) : U \in \mathcal{U}\}$  is a large open cover of  $X$ . Applying the groupability of  $\tilde{\mathcal{U}}$  for  $X$  and forgetting the  $X \setminus C$  part of the open sets shows the groupability of  $\mathcal{U}$  for  $C$ .

(2) Assume that  $f : X \rightarrow Y$  is a continuous surjection and that  $\mathcal{U}$  is a large open cover of  $Y$  (by open subsets of  $Y$ ). Then  $\mathcal{V} = \{f^{-1}[U] : U \in \mathcal{U}\}$  is a large open cover of  $X$ . By the assumption, there exists a groupable subcover  $\mathcal{W} \subseteq \mathcal{V}$  for  $X$ . It follows that  $\{U \in \mathcal{U} : f^{-1}[U] \in \mathcal{W}\}$  is a groupable cover of  $Y$ . ■

The following theorem tells us that *for our purposes*, it does not matter which notion of large covers we use (so that we can switch between the two notions at our convenience).

THEOREM 2. *Assume that  $X$  is a subspace of  $Y$  and  $(\overset{A_X}{\underset{A^{\text{gp}}_X}{X}})$  holds. Then each countable collection  $\mathcal{U}$  of open sets in  $Y$  which is a large cover of  $X$  is groupable for  $X$ .*

*Proof.* We will repeatedly use the following lemma.

LEMMA 3. *Assume that  $X$  is a subspace of  $Y$ , and  $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$  is a large open cover of  $X$  by open subsets of  $Y$ . Define an equivalence relation  $\sim$  on  $\mathbb{N}$  by*

$$n \sim m \quad \text{if} \quad X \cap U_n = X \cap U_m.$$

Let  $A = \{n : [n] \text{ is infinite}\}$ , and  $V = \bigcup_{n \in A} U_n$ . Then  $\{U_n : n \in A\}$  is a groupable cover of  $X \cap V$ , and  $\{U_n : n \notin A\}$  is a large cover of  $X \setminus V$  (by open subsets of  $Y$ ).

*Proof.* Define a partition of  $A$  as follows: Let  $[n_0], [n_1], \dots$  enumerate the elements of  $A/\sim$ . Let  $F_0$  contain the first element of  $[n_0]$ ,  $F_1$  contain the second element of  $[n_0]$  and the first element of  $[n_1]$ ,  $F_2$  contain the third element of  $[n_0]$ , the second element of  $[n_1]$  and the first element of  $[n_2]$ , etc. Fix  $x \in X \cap V$ , and let  $i$  be such that  $x \in U_{n_i}$ . Then for all but finitely many  $n$ , there exists  $k \in F_n \cap [n_i]$ , and therefore  $x \in U_k$ . This proves the first assertion.

Assume that  $X \not\subseteq V$ . As  $\mathcal{U}$  is a large cover of  $X \setminus V$  and for  $x \in X \setminus V$  and  $n \in A$ ,  $x \notin U_n$ , there must exist infinitely many  $n \notin A$  such that  $x \in U_n$ . ■

We now prove Theorem 2. Enumerate  $\mathcal{U}$  bijectively as  $\{U_n\}_{n \in \mathbb{N}}$ . We make the following definition by transfinite induction on  $\alpha < \aleph_1$  (and make sure that indeed it terminates at some  $\alpha < \aleph_1$ ). Carry out the following construction as long as  $A_\alpha$  is not empty.

- (1) *First step:* Set  $X_0 = X$ ,  $B_0 = \mathbb{N}$ , and  $V_0 = \emptyset$ .
- (2) *Successor step:* Assume that  $X_\alpha$ ,  $B_\alpha$ , and  $V_\alpha$  are defined, and  $\{U_n : n \in B_\alpha\}$  is a large cover of  $X_\alpha \setminus V_\alpha$ . Set  $X_{\alpha+1} = X_\alpha \setminus V_\alpha$ , and define an equivalence relation  $\sim_{\alpha+1}$  on  $B_\alpha$  by  $n \sim_{\alpha+1} m$  if  $X_{\alpha+1} \cap U_n = X_{\alpha+1} \cap U_m$ . Let  $A_{\alpha+1} = \{n \in B_\alpha : [n]_{\sim_{\alpha+1}} \text{ is infinite}\}$ ,  $B_{\alpha+1} = B_\alpha \setminus A_{\alpha+1}$ , and  $V_{\alpha+1} = \bigcup_{n \in A_{\alpha+1}} U_n$ . Use Lemma 3 to obtain a partition  $\{F_n^{\alpha+1}\}_{n \in \mathbb{N}}$  of  $A_{\alpha+1}$  into finite sets witnessing that  $\{U_n : n \in A_{\alpha+1}\}$  is a groupable cover of  $X_{\alpha+1} \cap V_{\alpha+1}$  (and  $\{U_n : n \in B_{\alpha+1}\}$  is a large cover of  $X_{\alpha+1} \setminus V_{\alpha+1}$ ).
- (3) *Limit step:* Assume that  $\alpha$  is a limit and the construction was carried out up to step  $\alpha$ . Set  $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ ,  $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ ,  $B_\alpha = \bigcap_{\beta < \alpha} B_\beta = \mathbb{N} \setminus A_\alpha$ , and  $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$ . For each  $x \in X_\alpha$  and each  $\beta < \alpha$ ,  $x \notin V_\beta$ , that is,  $\{n : x \in U_n\}$  is disjoint from  $A_\beta$ . Thus,  $\{n : x \in U_n\}$  is infinite, and is a subset of  $B_\alpha$ . In other words,  $\{U_n : n \in B_\alpha\}$  is a large cover of  $X_\alpha$ . Observe that in this case,  $X_\alpha$  is disjoint from  $V_\alpha$ .

As long as the construction continues,  $A_\alpha$  is not empty and therefore  $B_{\alpha+1}$  is a proper subset of  $B_\alpha$ . Thus, as  $B_0 \subseteq \mathbb{N}$ , the construction cannot continue for uncountably many steps. Let  $\alpha < \aleph_1$  be the step where the construction terminates (this can only happen when  $\alpha$  is a successor). Then  $A_\alpha$  is empty, therefore  $V_\alpha$  is empty, thus  $\{U_n : n \in B_\alpha\}$  is a large cover of  $X_\alpha$ . The definition of  $B_\alpha$  implies that in this case,  $\{U_n \cap X_\alpha : n \in B_\alpha\}$  is a large cover of  $X_\alpha$  by open subsets of  $X_\alpha$ . By the construction,  $X_\alpha$  is a closed subset of  $X$  (an intersection of closed sets). By Lemma 1,  $\{U_n \cap X_\alpha : n \in B_\alpha\}$  is

groupable for  $X_\alpha$ ; let  $\{F_n^{\alpha+1}\}_{n \in \mathbb{N}}$  be a partition of  $B_\alpha$  into finite sets that witnesses that.

The partitions  $\{F_n^{\beta+1}\}_{n \in \mathbb{N}}$  where  $\beta \leq \alpha$  form a countable family of partitions of disjoint subsets of  $\mathbb{N}$ . Relabel these partitions as  $\{\{G_n^m\}_{n \in \mathbb{N}} : m \in \mathbb{N}\}$ , and define a partition  $\{H_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets by

$$H_n = \bigcup_{\max\{i,j\}=n} G_j^i.$$

Observe that  $X \subseteq X_\alpha \cup \bigcup_{\beta < \alpha} (V_{\beta+1} \setminus V_\beta)$ , where each  $X \cap (V_{\beta+1} \setminus V_\beta)$  is taken care of by  $\{F_n^{\beta+1}\}_{n \in \mathbb{N}}$ , and  $X_\alpha$  is taken care of by  $\{F_n^{\alpha+1}\}_{n \in \mathbb{N}}$ . Hence, for each  $x \in X$  there exists  $m$  such that  $x \in \cup\{U_k : k \in G_n^m\} \subseteq \cup\{U_k : k \in H_n\}$  for all but finitely many  $n$ . This shows that  $\mathcal{U}$  is a groupable cover of  $X$ . ■

### 3. The main theorem

**THEOREM 4.** *For a separable and zero-dimensional metrizable space  $X$ , the following are equivalent:*

- (1)  $X$  has the Hurewicz property.
- (2) Every large open cover of  $X$  contains a groupable cover of  $X$ .
- (3) Every countable large open cover of  $X$  is groupable.

*Proof.* (2) $\Leftrightarrow$ (3). By Proposition 1.1 of [13], every large open cover of  $X$  contains a countable large open cover of  $X$ .

(1) $\Rightarrow$ (3). This is proved in [11, Lemma 3] and [6, Lemma 8].

(3) $\Rightarrow$ (1). We will prove this assertion by a sequence of small steps, using the results of the previous section.

The Baire space  ${}^{\mathbb{N}}\mathbb{N}$  of infinite sequences of natural numbers is equipped with the product topology (where the topology of  $\mathbb{N}$  is discrete). A quasiordering  $\leq^*$  is defined on the Baire space  ${}^{\mathbb{N}}\mathbb{N}$  by eventual dominance:

$$f \leq^* g \text{ if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

We say that a subset  $Y$  of  ${}^{\mathbb{N}}\mathbb{N}$  is *bounded* if there exists  $g$  in  ${}^{\mathbb{N}}\mathbb{N}$  such that for each  $f \in Y$ ,  $f \leq^* g$ . Otherwise, we say that  $Y$  is *unbounded*. According to a theorem of Hurewicz [3] (see also Reclaw [9]),  $X$  has the Hurewicz property if, and only if, each continuous image of  $X$  in  ${}^{\mathbb{N}}\mathbb{N}$  is bounded. Let  ${}^{\mathbb{N}}\nearrow\mathbb{N}$  denote the subspace of  ${}^{\mathbb{N}}\mathbb{N}$  consisting of the strictly increasing elements of  ${}^{\mathbb{N}}\mathbb{N}$ . The mapping from  ${}^{\mathbb{N}}\mathbb{N}$  to  ${}^{\mathbb{N}}\nearrow\mathbb{N}$  defined by

$$f(n) \mapsto g(n) = f(0) + \dots + f(n) + n$$

is a homeomorphism which preserves boundedness in both directions. Consequently, Hurewicz's theorem can be stated using  ${}^{\mathbb{N}}\nearrow\mathbb{N}$  instead of  ${}^{\mathbb{N}}\mathbb{N}$ .

For  $f, g \in {}^{\mathbb{N}}\nearrow\mathbb{N}$ , we say that  $f$  goes through the slalom defined by  $g$  if for all but finitely many  $n$ , there exists  $m$  such that  $f(m) \in [g(n), g(n+1))$ .

A subset  $Y$  of  $\mathbb{N}^{\nearrow}\mathbb{N}$  admits a slalom if there exists  $g \in \mathbb{N}^{\nearrow}\mathbb{N}$  such that each  $f \in Y$  goes through the slalom  $g$ .

LEMMA 5 (folklore). Assume that  $Y \subseteq \mathbb{N}^{\nearrow}\mathbb{N}$ . The following are equivalent:

- (1)  $Y$  is bounded.
- (2)  $Y$  admits a slalom.
- (3) There exists a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $f \in Y$  and all but finitely many  $n$ , there exists  $m$  such that  $f(m) \in F_n$ .

For completeness, we give a short proof.

*Proof.* (1) $\Rightarrow$ (2). Assume that  $g \in \mathbb{N}^{\nearrow}\mathbb{N}$  bounds  $Y$ . Define inductively  $h \in \mathbb{N}^{\nearrow}\mathbb{N}$  by

$$h(0) = g(0), \quad h(n + 1) = g(h(n)) + 1.$$

Then for each  $f \in Y$  and all but finitely many  $n$ ,  $h(n) \leq f(h(n)) \leq g(h(n)) < h(n + 1)$ , that is,  $f(h(n)) \in [h(n), h(n + 1))$ .

(2) $\Rightarrow$ (1). Assume that  $Y$  admits a slalom  $g$ . Let  $h$  be a function which eventually dominates all functions of the form  $f(n) = g(n_0 + n)$ ,  $n_0 \in \mathbb{N}$ . Let  $f$  be any element of  $Y$  and choose  $n_0$  such that for each  $n \geq n_0$ , there exists  $m$  such that  $f(m) \in [g(n), g(n + 1))$ . Choose  $m_0$  such that  $f(m_0) \in [g(n_0), g(n_0 + 1))$ . By induction on  $n$ , we find that  $(f(n) \leq) f(m_0 + n) \leq g(n_0 + 1 + n)$  for all  $n$ . For large enough  $n$ , we have  $g(n_0 + 1 + n) \leq h(n)$ , thus  $f \leq^* h$ .

Clearly (2) $\Rightarrow$ (3). We will show that (3) $\Rightarrow$ (2). Let  $\{F_n\}_{n \in \mathbb{N}}$  be as in (2). Define  $g \in \mathbb{N}^{\nearrow}\mathbb{N}$  as follows: Set  $g(0) = 0$ . Having defined  $g(0), \dots, g(n - 1)$ , let  $m$  be minimal such that  $F_m \cap [0, g(n - 1)) = \emptyset$ , and set  $g(n) = \max F_m + 1$ . Then for each  $n$  there exists  $F_m$  such that  $F_m \subseteq [g(n), g(n + 1))$ . Consequently,  $Y$  admits the slalom defined by  $g$ . ■

The Cantor space  $\{0, 1\}^{\mathbb{N}}$  is also equipped with the product topology. Identify  $P(\mathbb{N})$  with  $\{0, 1\}^{\mathbb{N}}$  by characteristic functions. The Rothberger space  $P_{\infty}(\mathbb{N})$  is the subspace of  $P(\mathbb{N})$  consisting of all infinite sets of natural numbers. The space  $\mathbb{N}^{\nearrow}\mathbb{N}$  is homeomorphic to  $P_{\infty}(\mathbb{N})$  by identifying each  $f \in \mathbb{N}^{\nearrow}\mathbb{N}$  with its image  $f[\mathbb{N}]$  (so that  $f$  is the increasing enumeration of  $f[\mathbb{N}]$ ).

Translating Lemma 5 into the language of  $P_{\infty}(\mathbb{N})$  and using Hurewicz’s theorem, we obtain the following characterization of the Hurewicz property in terms of continuous images in the Rothberger space.

LEMMA 6. For a separable and zero-dimensional metrizable space  $X$ , the following are equivalent:

- (1)  $X$  has the Hurewicz property.

- (2) For each continuous image  $Y$  of  $X$  in  $P_\infty(\mathbb{N})$  there exists  $g \in \mathbb{N} \nearrow \mathbb{N}$  such that for each  $y \in Y$ ,  $y \cap [g(n), g(n + 1)) \neq \emptyset$  for all but finitely many  $n$ .
- (3) For each continuous image  $Y$  of  $X$  in  $P_\infty(\mathbb{N})$  there exists a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $y \in Y$ ,  $y \cap F_n \neq \emptyset$  for all but finitely many  $n$ .

Assume that every countable large open cover of  $X$  is groupable. We will show that (3) of Lemma 6 holds. Let  $Y$  be a continuous image of  $X$  in  $P_\infty(\mathbb{N})$ . Then by Lemma 1,  $(\begin{smallmatrix} A_Y \\ A_Y^{\text{gp}} \end{smallmatrix})$  holds. Thus, by Theorem 2, every countable large open cover of  $P_\infty(\mathbb{N})$  is groupable as a cover of  $Y$ .

Let  $\mathcal{U} = \{O_n\}_{n \in \mathbb{N}}$ , where for each  $n$ ,

$$O_n = \{a \in P_\infty(\mathbb{N}) : n \in a\}.$$

Then  $\mathcal{U}$  is a large open cover of  $P_\infty(\mathbb{N})$ . Thus, there exists a partition  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  of  $\mathcal{U}$  into finite sets such that for each  $y \in Y$ ,  $y \in \cup \mathcal{F}_n$  for all but finitely many  $n$ . For each  $n$  set  $F_n = \{m : O_m \in \mathcal{F}_n\}$ . Then  $\{F_n\}_{n \in \mathbb{N}}$  is a partition of  $\mathbb{N}$  into finite sets. For each  $y \in Y$  and for all but finitely many  $n$ , there exists  $k$  such that  $y \in O_k \in \mathcal{F}_n$ , that is,  $k \in y \cap F_n$ , therefore  $y \cap F_n \neq \emptyset$ . ■

REMARK 7. A strengthening of the Hurewicz property for  $X$ , considering *countable Borel* covers instead of open covers, was given the following simple characterization in [12]:

For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of countable (large) Borel covers of  $X$ , there exist elements  $U_n \in \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that  $X \subseteq \bigcup_n \bigcap_{m > n} U_m$ .

(Notice that the analogous equivalence for the open case does not hold [4].) Using the same proof as in Theorem 4, we deduce that this property is also equivalent to requiring that every countable large Borel cover of  $X$  is groupable.

Forgetting about the topology and considering only countable covers, we get the following characterization of the minimal cardinality  $\mathfrak{b}$  of an unbounded family in the Baire space  $\mathbb{N}^\mathbb{N}$ . For a cardinal  $\kappa$ , denote by  $\Lambda_\kappa$  (respectively,  $\Lambda_\kappa^{\text{gp}}$ ) the collection of countable large (respectively, groupable) covers of  $\kappa$ .

COROLLARY 8. For an infinite cardinal  $\kappa$ , the following are equivalent:

- (1)  $\kappa < \mathfrak{b}$ .
- (2) Each subset of  $\mathbb{N} \nearrow \mathbb{N}$  of cardinality  $\kappa$  admits a slalom.
- (3) For each family  $Y \subseteq P_\infty(\mathbb{N})$  of cardinality  $\kappa$ , there exists a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $y \in Y$ ,  $y \cap F_n \neq \emptyset$  for all but finitely many  $n$ .
- (4)  $(\begin{smallmatrix} \Lambda_\kappa \\ \Lambda_\kappa^{\text{gp}} \end{smallmatrix})$  holds (i.e., every countable large cover of  $\kappa$  is groupable).

REMARK 9. The underlying combinatorics in this paper is similar to that appearing in Bartoszyński's characterization of  $\text{add}(\mathcal{N})$  (the minimal cardinality of a family of measure zero sets whose union is not a measure zero set) [1]. The equivalence (1) $\Leftrightarrow$ (2) in Corollary 8 is folklore. The equivalence of these with (4) seems to be new.

The only other covering property we know of which enjoys the possibility of considering subcovers of a given cover instead of selecting from a given sequence of covers is the Gerlits–Nagy  $\gamma$ -property. The proof for this fact is much easier—see [2].

### References

- [1] T. Bartoszyński, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. 281 (1984), 209–213.
- [2] J. Gerlits and Zs. Nagy, *Some properties of  $C(X)$ , I*, Topology Appl. 14 (1982), 151–161.
- [3] W. Hurewicz, *Über Folgen stetiger Funktionen*, Fund. Math. 9 (1927), 193–204.
- [4] W. Just, A. W. Miller, M. Scheepers, and P. Szeptycki, *The combinatorics of open covers II*, Topology Appl. 73 (1996), 241–266.
- [5] Lj. D. R. Kočinac and M. Scheepers, *Function spaces and a property of Reznichenko*, *ibid.* 123 (2002), 135–143.
- [6] —, —, *Combinatorics of open covers (VII): Groupability*, Fund. Math. 179 (2003), 131–155.
- [7] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsber. Wiener Akad. 133 (1924), 421–444.
- [8] A. Nowik, M. Scheepers, and T. Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic 63 (1998), 301–324.
- [9] I. Reclaw, *Every Lusin set is undetermined in the point-open game*, Fund. Math. 144 (1994), 43–54.
- [10] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. 69 (1996), 31–62.
- [11] —, *Finite powers of strong measure zero sets*, J. Symbolic Logic 64 (1999), 1295–1306.
- [12] M. Scheepers and B. Tsaban, *The combinatorics of Borel covers*, Topology Appl. 121 (2002), 357–382.
- [13] B. Tsaban, *The combinatorics of splittability*, Ann. Pure Appl. Logic, to appear (<http://arxiv.org/abs/math.LO/0212312>).

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