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Hindman's coloring theorem in arbitrary semigroups

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ABSTRACT

Hindman's Theorem asserts that, for each finite coloring of the natural numbers, there are distinct natural numbers $a_1, a_2, \ldots$ such that all of the sums $a_{i_1} + a_{i_2} + \cdots + a_{i_m}$ ($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color.

The celebrated Galvin–Glazer proof of Hindman's Theorem and a classification of semigroups due to Shevrin, imply together that, for each finite coloring of each infinite semigroup S, there are distinct elements $a_1, a_2, \ldots$ of S such that all but finitely many of the products $a_{i_1} a_{i_2} \cdots a_{i_m}$ ($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color.

Using these methods, we characterize the semigroups S such that, for each finite coloring of S, there is an infinite *subsemigroup* T of S, such that all but finitely many members of T have the same color.

Our characterization connects our study to a classical problem of Milliken, Burnside groups and Tarski Monsters. We also present an application of Ramsey's graph-coloring theorem to Shevrin's theory.

Contents

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1. The Galvin–Glazer–Hindman Theorem

A *finite coloring* of a set A is an assignment of one color to each element of A, where the set of possible colors is finite. In 1974, Hindman proved the following theorem, extending profoundly a result of Hilbert.

Theorem 1.1. *(See Hindman [9].) For each finite coloring of $\mathbb{N}$, there are $a_1, a_2, \ldots \in \mathbb{N}$ such that all sums $a_{i_1} + a_{i_2} + \cdots + a_{i_m}$ ($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color.*

In Hindman's Theorem 1.1, we may request that the elements $a_1, a_2, \ldots$ are *distinct*, by moving, if needed, to appropriate disjointly supported finite sums thereof. We consider here generalizations of Hindman's Theorem to *arbitrary* semigroups. Since we do not restrict attention to the abelian case, we usually use multiplicative notation. Let S be an infinite, finitely colored semigroup. Fix $s \in S$. The homomorphism $n \mapsto s^n$ induces a coloring of $\mathbb{N}$, and by Hindman's Theorem there are distinct $a_1, a_2, \ldots \in \mathbb{N}$ such that all elements

$$s^{a_{i_1} + \cdots + a_{i_m}} = s^{a_{i_1}} s^{a_{i_2}} \cdots s^{a_{i_m}}$$

($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color. Setting $s_n = s^{a_n}$ for all n, we have that all products $s_{i_1} s_{i_2} \cdots s_{i_m}$ ($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color. But, unlike Hindman's Theorem, the latter consequence may be trivial: If, for example, s is an idempotent (i.e., $s^2 = s$) then the reason for all products having the same color is that the elements $s_1, s_2, \ldots$ and the finite products thereof are all equal to s!

Since its publication, several alternative proofs for Hindman's Theorem were published. The most elegant and powerful one, due to Galvin and Glazer, was first published in Comfort's survey [6]. The Galvin–Glazer proof uses idempotents in the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$, and generalizes with little effort to a proof of the following theorem. (Knowledge of the Stone–Čech compactification is not required in the present paper.)

Say that a semigroup S is *moving* if it is infinite and, for each infinite $A \subseteq S$ and each finite $F \subseteq S$, there are $a_1, \ldots, a_k \in A$ such that

$$\{a_1 s, a_2 s, \ldots, a_k s\} \nsubseteq F$$

for all but finitely many $s \in S$. Every right cancellative infinite semigroup is moving. Also, if left multiplication in S is finite-to-one (in particular, if S is left cancellative), then S is moving.

Theorem 1.2 *(Galvin–Glazer–Hindman). Let S be a moving semigroup. For each finite coloring of S, there are* distinct *$a_1, a_2, \ldots \in S$ such that all products $a_{i_1} a_{i_2} \cdots a_{i_m}$ ($m \geqslant 1$, $i_1 < i_2 < \cdots < i_m$) have the same color.*

Our purpose is to generalize the Galvin–Glazer–Hindman Theorem 1.2 to *arbitrary* infinite semigroups S, and to understand the limitations on such generalizations. We also consider stronger forms of this theorem.

Remark 1.3 *(Attribution).* Theorem 1.2, which we attribute to Galvin, Glazer and Hindman, is implicit in Section 4.3 of Hindman and Strauss's monograph [10]. There, it is proved that S is moving if, and only if, the Stone–Čech remainder $\beta S \setminus S$ is a subsemigroup of βS. It follows that $\beta S \setminus S$ contains an idempotent, and thus, by the standard Galvin–Glazer proof of Hindman's Theorem, there are distinct $a_1, a_2, \ldots \in S$ as required in Theorem 1.2.

2. Hindman's Theorem everywhere

As is, the Galvin–Glazer–Hindman Theorem 1.2 does not generalize to arbitrary semigroups: Consider the following example.

Example 2.1. Let $k \in \mathbb{N}$. Let S be the commutative semigroup

$$\{0, 1, \ldots, k-1\} \cup k\mathbb{N} + 1,$$

with the operation of addition modulo k. Assign to each $a \in S$ the color $a \bmod k$. For all distinct $a_1, a_2, \ldots \in S$, we may, by thinning out if necessary, assume that they are all in $k\mathbb{N} + 1$. Consequently, for each $i < k$, $a_1 + \cdots + a_i = i$, whose color is i. In other words, all colors $i < k$ are obtained when considering all sums of distinct elements from $\{a_1, a_2, \ldots\}$.

Thus, we must allow an unbounded finite number of exceptions. We will soon see that this is the only obstruction to generalizing the Galvin–Glazer–Hindman Theorem 1.2 to arbitrary semigroups.

We use Shevrin's classification of semigroups. A semigroup S is *periodic* if $\langle s\rangle$ is finite for all $s \in S$, or equivalently, if $\mathbb{N} \nleq S$. A semigroup S is *right* (*left*) *zero* if $ab = b$ ($ab = a$) for all $a, b \in S$.

Theorem 2.2. *(See Shevrin [13].) Every infinite semigroup has a subsemigroup of one of the following types:*

(1) $(\mathbb{N}, +)$.
(2) *An infinite periodic group.*
(3) *An infinite right zero or left zero semigroup.*
(4) $(\mathbb{N}, \vee)$, *where* $m \vee n := \max\{m, n\}$.
(5) $(\mathbb{N}, \wedge)$, *where* $m \wedge n := \min\{m, n\}$.
(6) *An infinite semigroup S with S^2 finite.*
(7) *The fan semilattice* $(\mathbb{N}, \wedge)$, *with* $m \wedge n = 1$ *for distinct* m, n.

Shevrin's Theorem is stated in [13] in a finer form, replacing (6) with a parameterized list of concrete semigroups. We will return to this in Section 5.

Theorem 2.3. *Let S be an infinite semigroup. For each finite coloring of S, there are* distinct $a_1, a_2, \ldots \in S$, *and a finite subset F of the* (*infinite*) *set of finite products*

$$\mathrm{FP}(a_1, a_2, \ldots) = \{a_{i_1} a_{i_2} \cdots a_{i_m} \colon m \geqslant 1,\ i_1 < i_2 < \cdots < i_m\},$$

such that all elements of $\mathrm{FP}(a_1, a_2, \ldots) \setminus F$ *have the same color.*

Proof. It suffices to show that every infinite semigroup has a subsemigroup satisfying the assertion of the theorem. Apply Shevrin's Theorem 2.2. The subsemigroups in cases (1)–(5) are all moving (!), and thus the Galvin–Glazer–Hindman Theorem 1.2 applies there.

In the remaining cases (6)–(7), let T be the corresponding infinite subsemigroup. By the pigeonhole principle, there are distinct $a_1, a_2, \ldots \in T$, sharing the same color. Then

$$\mathrm{FP}(a_1, a_2, \ldots) \subseteq \{a_1, a_2, \ldots\} \cup F,$$

where F is T^2 in case (6), and $\{1\}$ in case (7), and thus all elements of $\mathrm{FP}(a_1, a_2, \ldots) \setminus F$ have the same color. □

3. Infinite almost-monochromatic subsemigroups

Definition 3.1. A colored set A is *monochromatic* if all members of A have the same color. A is *almost-monochromatic* if all but finitely many members of A have the same color.

For which semigroups S is it the case that, for each finite coloring of S, there is an infinite almost-monochromatic subsemigroup of S? We begin with two easy examples.

Let $\mathbb{Z}_2$ be the two-element abelian group. The *direct sum* $\bigoplus_n \mathbb{Z}_2$ is the additive abelian group of all finitely supported elements of $\mathbb{Z}_2^{\mathbb{N}}$, with pointwise addition. In other words, $\bigoplus_n \mathbb{Z}_2$ is the group structure of the countably-infinite-dimensional vector space over the two-element field.

Lemma 3.2. *For each finite coloring of $\bigoplus_n \mathbb{Z}_2$, there is an infinite subgroup H of $\bigoplus_n \mathbb{Z}_2$ with $H \setminus \{0\}$ monochromatic.*

Proof. This follows from the Galvin–Glazer–Hindman Theorem 1.2, since every group is a moving semigroup, and in the group $\bigoplus_n \mathbb{Z}_2$,

$$\langle a_1, a_2, \ldots \rangle = \{a_{i_1} + a_{i_2} + \cdots + a_{i_m} : m \geqslant 1,\ i_1 < i_2 < \cdots < i_m\} \cup \{0\}. \qquad \square$$

Definition 3.3. A semigroup S is *synchronizing* if $ab \in \{a, b\}$ for all $a, b \in S$. It is *finitely synchronizing* if there is a finite $F \subseteq S$ such that $ab \in \{a, b\} \cup F$ for all $a, b \in S$.

Our second example is the class of infinite, finitely synchronizing semigroups.

Lemma 3.4. *Let S be an infinite, finitely synchronizing semigroup. For each finite coloring of S, there is an infinite almost-monochromatic subsemigroup of S.*

Proof. By the pigeon-hole principle, there are distinct $a_1, a_2, \ldots \in S$, sharing the same color. Let F be a finite subset of S such that $ab \in \{a, b\} \cup F$ for all $a, b \in S$. As

$$\langle a_1, a_2, \ldots \rangle \subseteq \{a_1, a_2, \ldots\} \cup F,$$

$\langle a_1, a_2, \ldots \rangle$ is almost-monochromatic. $\square$

The main result of this section is that the above two easy examples provide a complete answer to our question. A 2-*coloring* of a set A is a coloring of A in two colors.

Theorem 3.5. *The following are equivalent for semigroups S:*

(1) *For each finite coloring of S, there is an infinite almost-monochromatic subsemigroup of S.*
(2) *For each 2-coloring of S, there is an infinite almost-monochromatic subsemigroup of S.*
(3) *At least one of the following assertions holds:*
   (a) *S has an infinite, finitely synchronizing subsemigroup.*
   (b) *$\bigoplus_n \mathbb{Z}_2 \leqslant S$.*

Item (3)(a) of the theorem may be replaced by an explicit list of semigroups, namely, the semigroups of types (3)–(7) in Shevrin's Theorem 2.2. Recall that item (6) can be replaced by a parameterized list of concrete semigroups—see Theorem 5.3 below. Thus, our characterization is completely explicit.

The implication (3) ⇒ (1) in Theorem 3.5 is clear. Indeed, if S has a subsemigroup T such that for any finite coloring of T, T contains an infinite almost-monochromatic subsemigroup, then the same holds for S. The implication (1) ⇒ (2) is clear as well. The remainder of this section constitutes a proof of the implication (2) ⇒ (3).

Lemma 3.6 *(Folklore). Let G be an infinite group such that all elements of $G \setminus \{e\}$ have order 2. Then G is isomorphic to $\bigoplus_{\alpha \in I} \mathbb{Z}_2$, where I is an index set of cardinality $|G|$. In particular, $\bigoplus_n \mathbb{Z}_2 \leqslant G$.*

Proof. G is commutative: $[g,h] = ghg^{-1}h^{-1} = (gh)^2 = e$ for all $g,h \in G$. Thus, we may use additive notation for G, so that $v + v = 0$ for each $v \in G$, and G is a vector space over the two-element field, necessarily of dimension $|G|$. In other words, G is isomorphic to $\bigoplus_{\alpha \in I} \mathbb{Z}_2$. □

Lemma 3.7. *Let G be a group. There is a 2-coloring of the elements of G of finite order greater than 2 such that, for each coloring of G extending it and each infinite periodic almost-monochromatic subgroup $H \leqslant G$, $\bigoplus_n \mathbb{Z}_2 \leqslant H$.*

Proof. For each $g \in G$ of finite order greater than 2, color g and g^{-1} differently. Let H be an infinite periodic almost-monochromatic subgroup of G. If there are infinitely many $h \in H$ with $h^2 \neq e$, then there are infinitely many such elements of the same color. But then their inverses, which have the opposite color, also belong to H; a contradiction. Thus, all but finitely many members of H have order 2. Let F be the set of elements of order $\neq 2$ in H.

Pick $h_1 \in H \setminus F$. Then $\langle h_1 \rangle = \{h_1, e\}$ is finite. For $n > 1$, assume inductively that all elements of the subgroup $K := \langle h_1, \dots, h_{n-1} \rangle$ of H have order $\leqslant 2$. Then K is commutative and finite. Pick

$$h_n \in H \setminus \bigcup_{h \in K} Fh.$$

Then $h_n \notin K \cup F$ and $h_n h \notin F$ for all $h \in K$. Consequently, the order of h_n is 2, and for each $h \in K$, the order of $h_n h$ is 2. It follows that $h_n h = h h_n$, and thus $\langle h_1, \dots, h_n \rangle$ is commutative, finite, and all of its elements have order $\leqslant 2$.

Completing the induction, we have by Lemma 3.6 that $\langle h_1, h_2, \dots \rangle$ is isomorphic to $\bigoplus_n \mathbb{Z}_2$. □

Lemma 3.8 *(Folklore). There is a 2-coloring of $\mathbb{N}$ with no infinite almost-monochromatic subsemigroup.*

Proof. Consider the coloring

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ \dots,$$

where the lengths of the intervals of elements of identical colors are $1, 2, 3, \dots$. For each $n \in \mathbb{N}$, $\langle n \rangle$ intersects every monochromatic interval of length $\geqslant n$. □

For a semigroup S and an idempotent $e \in S$, let $G(e)$ be the maximal subgroup of the semigroup S containing the idempotent e. As groups have exactly one idempotent, $G(e_1) \cap G(e_2) = \emptyset$ for all distinct idempotents $e_1, e_2 \in S$.

True Color Lemma 3.9. *For each semigroup S, there is a 2-coloring of S such that:*

(1) *Every almost-monochromatic subsemigroup of S is periodic; and*
(2) *Every infinite almost-monochromatic subgroup of S contains $\bigoplus_n \mathbb{Z}_2$ as a subgroup.*

Proof. An *orbit* in S is a subset of the form $\langle s \rangle$ for some $s \in S$. If there are infinite orbits in S, use Zorn's Lemma to obtain a maximal family $\mathcal{F}$ of disjoint infinite orbits in S. If there are none, let $\mathcal{F} = \emptyset$. For each $\langle s \rangle \in \mathcal{F}$, $\langle s \rangle$ is isomorphic to $\mathbb{N}$. Use Lemma 3.8 to obtain, for each $\langle s \rangle \in \mathcal{F}$, a coloring of $\langle s \rangle$ in red and green, such that $\langle s \rangle$ has no almost-monochromatic subsemigroup.

Let e be an idempotent of S. The elements of finite order greater than 2 in $G(e)$ do not belong to an infinite orbit, and are thus not colored yet. Color these elements in red and green, as in Lemma 3.7. As the groups $G(e)$ are disjoint for distinct idempotents, this can be done for all idempotents.

Extend our partial 2-coloring to an arbitrary 2-coloring of S.

(1) Let T be a non-periodic subsemigroup of S. Pick $t \in T$ with $\langle t \rangle$ infinite. By the maximality of $\mathcal{F}$, $\langle t \rangle$ intersects some $\langle s \rangle \in \mathcal{F}$. Let n be such that $t^n \in \langle s \rangle$. Then the subsemigroup $\langle t^n \rangle$ of $\langle s \rangle$ is not almost-monochromatic. In particular, T is not almost-monochromatic.

(2) Let G be an infinite almost-monochromatic subgroup of S. By (1), G is periodic. Let e be the idempotent of G. Then $G \leqslant G(e)$, and by Lemma 3.7, $\bigoplus_n \mathbb{Z}_2 \leqslant G$. □

Proof of Theorem 3.5. Assume that, for each 2-coloring of S, there is an infinite almost-monochromatic subsemigroup of S. Color S as in the True Color Lemma 3.9. Let T be an almost-monochromatic subsemigroup of S. By the True Color Lemma, T is periodic.

If T has no infinite subgroup, then cases (1) and (2) in Shevrin's Theorem 2.2 are excluded. As each of the semigroups in the remaining cases of Shevrin's Theorem is finitely synchronizing, S has an infinite, finitely synchronizing subsemigroup.

And if T has an infinite subgroup, G, then by the True Color Lemma, $\bigoplus_n \mathbb{Z}_2 \leqslant G$. □

The case of *groups* is of independent interest.

Theorem 3.10. *The following are equivalent for groups G:*

(1) *For each finite coloring of G, there is an infinite almost-monochromatic subgroup of G.*
(2) *For each finite coloring of G, there is an infinite almost-monochromatic subsemigroup of G.*
(3) *For each 2-coloring of G, there is an infinite almost-monochromatic subgroup of G.*
(4) *For each 2-coloring of G, there is an infinite almost-monochromatic subsemigroup of G.*
(5) $\bigoplus_n \mathbb{Z}_2 \leqslant G$.

Proof. Clearly, the implications $(1 \Rightarrow 2)$, $(1 \Rightarrow 3)$, $(2 \Rightarrow 4)$ and $(3 \Rightarrow 4)$ hold.

$(4 \Rightarrow 5)$ Apply Theorem 3.5 to the group G. If T is an infinite, finitely synchronizing subsemigroup of G, then T is a periodic subsemigroup of a group, and thus a group. But infinite groups cannot be finitely synchronizing. Indeed, let F be a finite subset of G, $a \in G \setminus \{e\}$. Since left multiplication by a is injective, there is $b \in G \setminus \{e\}$ such that $ab \notin F$. As $a, b \neq e$, $ab \notin \{a, b\}$, and thus $a, b \notin \{a, b\} \cup F$. Consequently, we are in case (3)(b) of Theorem 3.5, that is, $\bigoplus_n \mathbb{Z}_2 \leqslant G$.

$(5 \Rightarrow 1)$ Lemma 3.2. □

4. Unordered products

For distinct $a_1, a_2, \ldots \in S$, let

$$\widehat{\mathrm{FP}}(a_1, a_2, \ldots) = \{a_{i_1} a_{i_2} \cdots a_{i_m}\colon m \geqslant 1,\ i_1, i_2, \ldots, i_m \text{ are distinct}\}.$$

We apply the information gathered in the previous sections to the following question: Let S be a prescribed infinite semigroup. Is it true that, for each finite coloring of S, there are distinct $a_1, a_2, \ldots \in S$ such that $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ is almost-monochromatic?

To see that the new question is different than the one studied in the previous section, note that, by Hindman's Theorem, for each finite coloring of $\mathbb{N}$, there are distinct $a_1, a_2, \ldots \in \mathbb{N}$ such that the set

$$\widehat{\mathrm{FP}}(a_1, a_2, \ldots) = \{a_{i_1} + a_{i_2} + \cdots + a_{i_m}\colon m \geqslant 1,\ i_1, i_2, \ldots, i_m \text{ are distinct}\}$$

is monochromatic, but there is a 2-coloring of $\mathbb{N}$ with no infinite almost-monochromatic subsemigroup (Lemma 3.8).

Theorem 4.1. *Assume that S has an infinite subsemigroup with no infinite, finitely generated, periodic subgroup. For each finite coloring of S, there are distinct $a_1, a_2, \ldots \in S$ with $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ almost-monochromatic.*

Proof. Assume that the theorem fails for S. Then $(\mathbb{N}, +)$ is not a subsemigroup of S. By moving to a subsemigroup of S, if needed, we may assume that S is an infinite semigroup with no infinite, finitely generated, periodic subgroup.

By Theorem 3.5, S does not contain an infinite finitely synchronizing subsemigroup. Thus, by Shevrin's Theorem 2.2, S has an infinite periodic subgroup G.

If G is locally finite, then it contains an infinite *abelian* group H [8]. As groups are moving, by the Galvin–Glazer–Hindman Theorem 1.2 there are distinct $a_1, a_2, \ldots \in H$ such that $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ is monochromatic; a contradiction.

Thus, G is not locally finite. Let $F \subseteq G$ be a finite set with $H := \langle F \rangle$ infinite. Then H is an infinite, finitely generated, periodic subgroup of S; a contradiction. □

The condition on S in Theorem 4.1 is quite mild: The 1902 *Burnside Problem* [5], that asked whether there is, at all, an infinite finitely generated periodic group, was only answered (in the affirmative) in 1964 [7].

The question whether the condition in Theorem 4.1 can be eliminated is equivalent to a 1978 problem of Milliken.

Proposition 4.2. *Let G be an infinite, finitely colored group. Assume that $a_1, a_2, \ldots \in G$ are distinct elements such that $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ is almost-monochromatic. Then there is a subsequence $a_{i_1}, a_{i_2}, \ldots$ of $a_1, a_2, \ldots$ such that $\widehat{\mathrm{FP}}(a_{i_1}, a_{i_2}, \ldots)$ is monochromatic.*

Proof. Let F be the finite set of elements of $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ having exceptional colors. Pick $a_{i_1} \in \{a_1, a_2, \ldots\} \setminus F$, and for $n > 1$, let P be the set of all products of at most $n-1$ distinct elements from $\{a_{i_1}, \ldots, a_{i_{n-1}}\}$, including also e. Pick $a_{i_n} \in \{a_1, a_2, \ldots\} \setminus P^{-1}FP^{-1}$, with $i_n > i_{n-1}$. □

Problem 4.3. *(See Milliken [11].) Is it true that, for each infinite, finitely colored group G, there are distinct $a_1, a_2, \ldots \in G$ such that $\widehat{\mathrm{FP}}(a_1, a_2, \ldots)$ is monochromatic?*

In 1968, Novikov and Adian [3] proved that, for each $m \geqslant 2$ and each large enough odd n, the *Burnside group*

$$G = \langle x_1, \ldots, x_m \colon x^n = 1 \rangle$$

(where $x^n = 1$ for all $x \in G$) is infinite (cf. Adian [1]). As was already noted by Milliken [11], for large enough odd n these groups have no infinite abelian subgroups [4], and thus the Galvin–Glazer–Hindman Theorem does not apply to them directly.

A group G is a *Tarski Monster* if, for some prime number p, all proper subgroups of G have cardinality p. Tarski Monsters exist for all large enough primes p (Olshanskii [12]; cf. Adian and Lysënok [2]). Clearly, Tarsky Monsters do not have infinite abelian subgroups. Thus, it may be possible to address Milliken's problem by finding the "true color" of some Tarski Monster...

5. A semigroup structure theorem of Shevrin, via Ramsey's Theorem

In the previous sections, we applied Shevrin's theory to coloring theory. We conclude with an application in the converse direction.

The following assertion is made in [13]. For completeness, we give a proof.

Lemma 5.1. *(See Shevrin [13].) Let S be a semigroup generated by A, such that, for some natural numbers $h > 1$ and d:*

(a) $abc = def$ *for all* $a, b, c, d, e, f \in A$;
(b) $a^h = a^{h+d}$ *for all* $a \in A$.

Then:

(1) $S^3 = \langle a \rangle^3$ *for each* $a \in A$.
(2) S^3 *is finite.*
(3) *There is a unique idempotent* $e \in S$.
(4) *For all* $a, b \in A$, $ae = be$.

Proof. (1) Each $s \in S^3$ is a product of $k \geqslant 3$ elements of A. Applying (a) repeatedly, we conclude that $s = a^k$.

(2) Fix $a \in A$. By (b), $\langle a \rangle$ is finite. Apply (1).

(3) Fix $a \in A$. By (b), $\langle a \rangle$ is finite, and thus there is an idempotent $e = a^k$ in $\langle a \rangle$. Let $s \in S$ be an idempotent. By (1), $s = s^3 \in \langle a \rangle$, say $s = a^m$. Then $s = s^k = a^{mk} = e^m = e$.

(4) Let $a, b \in A$. By (1), $e = e^3 \in \langle a \rangle^3$, and hence $e = a^k$ for some $k \geqslant 3$. By (a), $ae = a^{k+1} = a^3 a^{k-2} = ba^2 a^{k-2} = ba^k = be$. □

Following Shevrin [13], say that a semigroup S is of *type* $[h, d]$ for $h, d \in \mathbb{N}$ with $h > 1$, if S is generated by a countably infinite alphabet $x_1, x_2, \ldots$, with the following defining relations:

(HD1) $x_i^2 = x_1^2$ for all i;
(HD2) $x_i x_j = x_1 x_2$ and $x_j x_i = x_2 x_1$ for all $i < j$;
(HD3) $x_i x_j x_k = x_1^3$ for all i, j, k;
(HD4) $x_i^h = x_i^{h+d}$ for all i;

and possibly by additional relations, equating some or all of the elements: x_1^2, $x_1 x_2$, $x_2 x_1$, $(x_1 e)^2$, where e is the unique idempotent of S (Lemma 5.1(3)).

Shevrin proves in [13] a finer version of Theorem 2.2, where "An infinite semigroup S with S^2 finite" is replaced by "A semigroup of type $[h, d]$." In the course of his proof, however, he essentially proves the equivalence of these two versions of Theorem 2.2. We give a short, complete proof using Ramsey's celebrated coloring theorem. Ramsey's Theorem asserts that, for each finite coloring of the edges of an infinite complete graph, there is an infinite complete subgraph with all edges of the same color.

We first treat the easier implication of Shevrin's Theorem.

Proposition 5.2. *(See Shevrin [13].) Let S be a semigroup of type $[h, d]$. Then S is infinite, and S^2 is finite.*

Proof. As all of the words in the defining relations have more than one letter, there are no relations applicable to a single letter. Consequently, all letters of A are distinct in S, and S is infinite.

By Lemma 5.1(2), S^3 is finite. By the defining relations, $S^2 \setminus S^3 \subseteq \{x_1^2, x_1 x_2, x_2 x_1\}$. Thus, S^2 is finite. □

Theorem 5.3. *(See Shevrin [13].) Let S be an infinite semigroup with S^2 finite. Then S has a subsemigroup of type $[h, d]$, for some natural numbers $h > 1$ and d.*

Proof. As $S^3 \subseteq S^2$, S^3 is finite too. Pick distinct elements $a_1, a_2, \ldots \in S \setminus S^2$. Consider the complete infinite graph with vertex set $V = \{a_1, a_2, \ldots\}$. Think of the finite set $S^3 \times S^2 \times S^2 \times S^2$ as a set of colors, and define a finite coloring of the edges of our graph,

$$c : [V]^2 \to S^3 \times S^2 \times S^2 \times S^2,$$

by

$$c\big(\{a_i, a_j\}\big) := \big(a_i^3, a_i^2, a_i a_j, a_j a_i\big)$$

for all $i < j$. By Ramsey's Theorem, there are $i_1 < i_2 < \cdots$ such that all edges among the vertices in the set $\{a_{i_1}, a_{i_2}, \ldots\}$ have the same color. Denote $b_n = a_{i_n}$ for all n.

Let $1 \leqslant i < j$. Then

$$\big(b_i^3, b_i^2, b_i b_j, b_j b_i\big) = c\big(\{b_i, b_j\}\big) = c\big(\{b_2, b_3\}\big) = \big(b_2^3, b_2^2, b_2 b_3, b_3 b_2\big).$$

Hence, for all $1 \leqslant i < j$,

$$\begin{aligned} b_i^3 &= b_2^3; \\ b_i^2 &= b_2^2; \\ b_i b_j &= b_2 b_3; \quad \text{and} \\ b_j b_i &= b_3 b_2. \end{aligned}$$

We claim that the subsemigroup $T = \langle b_2, b_3, \ldots\rangle$ is of type $[h, d]$ for some $h > 1$ and d, with respect to the alphabet $b_2, b_3, \ldots$.

The elements $b_2, b_3, \ldots$ are distinct, being a subsequence of the sequence $a_1, a_2, \ldots$ of distinct elements.

We have already proved that relations (HD1) and (HD2) hold.

(HD3) Fix $k \geqslant 2$. If $i < j$, then $b_i b_j = b_2 b_3 = b_1 b_k$, and thus

$$b_i b_j b_k = b_1 b_k b_k = b_1 b_2^2 = b_1^3 = b_2^3.$$

If $i > j$, then

$$\begin{aligned} b_i b_j b_k &= (b_3 b_2) b_k = (b_{k+1} b_k) b_k = b_{k+1} b_k^2 \\ &= b_{k+1} b_2^2 = b_{k+1} b_{k+1}^2 = b_{k+1}^3 = b_2^3. \end{aligned}$$

If $i = j$, then

$$b_i b_j b_k = (b_i b_i) b_k = b_2^2 b_k = b_k^2 b_k = b_k^3 = b_2^3.$$

(HD4) Denote $b = b_2$. As S^2 is finite, so is $\langle b\rangle$. Take minimal h and d such that $b^h = b^{h+d}$. As $b \in S \setminus S^2$, $h > 1$. Thus, for all $i \geqslant 2$, we have by (HD1) and (HD3) that

$$b_i^h = b^h = b^{h+d} = b_i^{h+d}.$$

By (HD1)–(HD4),

$$T = \{b_2, b_3, \ldots\} \cup \big\{b_2^2, b_2 b_3, b_3 b_2\big\} \cup \big\{b^3, \ldots, b^{h+d-1}\big\},$$

where $b = b_2$. (In the case $h = 2$ and $d = 1$, the rightmost set in this union is empty.)

By Lemma 5.1, (HD3) and (HD4), there is a unique idempotent e in T. It thus remains to show that no additional equalities, except perhaps ones among b_2^2, $b_2 b_3$, $b_3 b_2$, and $(be)^2$, hold.

We already observed that the elements $b_2, b_3, \ldots$ are distinct, and by their choice, do not belong to S^2. Thus, equalities may only hold among members of the set $\{b_2^2, b_2 b_3, b_3 b_2\} \cup \{b^3, b^4, \ldots, b^{h+d-1}\}$.

In the case $h=2$ and $d=1$, the rightmost set is empty, and we are done. Consider the other cases. By the minimality of h and d, the elements $b=b_2, b^2=b_2^2, b^3, \dots, b^{h+d-1}$ are distinct, and

$$G := \{b^h, b^{h+1}, \dots, b^{h+d-1}\}$$

is a group. The idempotent element of G must be the unique idempotent e of S.

Thus, equalities may only hold among b_2b_3, b_3b_2 (which is fine), or be of the form $b^n = b_2b_3$ or $b^m = b_3b_2$, for some (necessarily unique) $2 \leqslant n, m \leqslant h+d-1$. It suffices to show that the former case is equivalent to $b_2b_3 = b_2^2$ or to $b_2b_3 = (be)^2$ and the latter to $b_3b_2 = b_2^2$ or to $b_3b_2 = (be)^2$.

We prove the assertion for b_2b_3; the other proof being identical.

As $e \in G$, $be \in G$ and thus so is $(be)^2$. Thus, if $b_2b_3 = (be)^2$, then there is $2 \leqslant h \leqslant n \leqslant h+d-1$ with $b_2b_3 = b^n$.

For the other direction, if $b^n = b_2b_3$ for $n > 2$ then

$$b^{n+1} = b^n b = b_2b_3b = b^3,$$

and therefore $h \leqslant 3 \leqslant n$. Thus, $b^n \in G$. As $e \in G \subseteq \langle b \rangle$, $be = eb$. Then

$$b^n(be) = b^{n+1}e = b^3e = b^3e^3 = (be)^2(be).$$

As $b^n, be \in G$, this implies that $b_2b_3 = b^n = (be)^2$, as required. □

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