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POINT-COFINITE COVERS IN THE LAVER MODEL

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Abstract. Let $\mathsf{S}_1(\Gamma,\Gamma)$ be the statement: For each sequence of point-cofinite open covers, one can pick one element from each cover and obtain a point-cofinite cover. $\mathfrak{b}$ is the minimal cardinality of a set of reals not satisfying $\mathsf{S}_1(\Gamma,\Gamma)$. We prove the following assertions:

(1) If there is an unbounded tower, then there are sets of reals of cardinality $\mathfrak{b}$ satisfying $\mathsf{S}_1(\Gamma,\Gamma)$.
(2) It is consistent that all sets of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ have cardinality smaller than $\mathfrak{b}$.

These results can also be formulated as dealing with Arhangel'skiĭ's property α_2 for spaces of continuous real-valued functions.

The main technical result is that in Laver's model, each set of reals of cardinality $\mathfrak{b}$ has an unbounded Borel image in the Baire space ω^ω.

1. Background

Let P be a nontrivial property of sets of reals. The *critical cardinality* of P, denoted $\mathrm{non}(P)$, is the minimal cardinality of a set of reals not satisfying P. A natural question is whether there is a set of reals of cardinality at least $\mathrm{non}(P)$, which satisfies P, i.e., a *nontrivial* example.

We consider the following property. Let X be a set of reals. $\mathcal{U}$ is a *point-cofinite* cover of X if $\mathcal{U}$ is infinite, and for each $x\in X$, $\{U\in\mathcal{U} : x\in U\}$ is a cofinite subset of $\mathcal{U}$.[1] Having X fixed in the background, let Γ be the family of all point-cofinite *open* covers of X. The following properties were introduced by Hurewicz [8], Tsaban [19], and Scheepers [15], respectively.

$\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$**:** For all $\mathcal{U}_0,\mathcal{U}_1,\dots\in\Gamma$, none containing a finite subcover, there are finite $\mathcal{F}_0\subseteq\mathcal{U}_0,\mathcal{F}_1\subseteq\mathcal{U}_1,\dots$ such that $\{\bigcup\mathcal{F}_n : n\in\omega\}\in\Gamma$.

$\mathsf{U}_2(\Gamma,\Gamma)$**:** For all $\mathcal{U}_0,\mathcal{U}_1,\dots\in\Gamma$, there are $\mathcal{F}_0\subseteq\mathcal{U}_0,\mathcal{F}_1\subseteq\mathcal{U}_1,\dots$ such that $|\mathcal{F}_n|=2$ for all n, and $\{\bigcup\mathcal{F}_n : n\in\omega\}\in\Gamma$.

$\mathsf{S}_1(\Gamma,\Gamma)$**:** For all $\mathcal{U}_0,\mathcal{U}_1,\dots\in\Gamma$, there are $U_0\in\mathcal{U}_0,U_1\in\mathcal{U}_1,\dots$ such that $\{U_n : n\in\omega\}\in\Gamma$.

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[1] Historically, point-cofinite covers were named γ-*covers*, since they are related to a property numbered γ in a list from α to ϵ in the seminal paper [7] of Gerlits and Nagy.

Clearly, $\mathsf{S}_1(\Gamma,\Gamma)$ implies $\mathsf{U}_2(\Gamma,\Gamma)$, which in turn implies $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$. None of these implications is reversible in ZFC [19]. The critical cardinality of all three properties is $\mathfrak{b}$ [9].[2]

Bartoszyński and Shelah [1] proved that there are, provably in ZFC, totally imperfect sets of reals of cardinality $\mathfrak{b}$ satisfying the Hurewicz property $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$. Tsaban proved the same assertion for $\mathsf{U}_2(\Gamma,\Gamma)$ [19]. These sets satisfy $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$ in all finite powers [2].

We show that in order to obtain similar results for $\mathsf{S}_1(\Gamma,\Gamma)$, hypotheses beyond ZFC are necessary.

2. Constructions

We show that certain weak (but not provable in ZFC) hypotheses suffice to have nontrivial $\mathsf{S}_1(\Gamma,\Gamma)$ sets, even ones which possess this property in all finite powers.

Definition 2.1. A *tower* of cardinality κ is a set $T\subseteq[\omega]^\omega$ which can be enumerated bijectively as $\{x_\alpha:\alpha<\kappa\}$, such that for all $\alpha<\beta<\kappa$, $x_\beta\subseteq^* x_\alpha$.

A set $T\subseteq[\omega]^\omega$ is *unbounded* if the set of its enumeration functions is unbounded; i.e., for any $g\in\omega^\omega$ there is an $x\in T$ such that for infinitely many n, $g(n)$ is less than the n-th element of x.

Scheepers [16] proved that if $\mathfrak{t}=\mathfrak{b}$, then there is a set of reals of cardinality $\mathfrak{b}$ satisfying $\mathsf{S}_1(\Gamma,\Gamma)$. If $\mathfrak{t}=\mathfrak{b}$, then there is an unbounded tower of cardinality $\mathfrak{b}$, but the latter assumption is weaker.

Lemma 2.2 (folklore)**.** *If $\mathfrak{b}<\mathfrak{d}$, then there is an unbounded tower of cardinality $\mathfrak{b}$.*

Proof. Let $B=\{b_\alpha:\alpha<\mathfrak{b}\}\subseteq\omega^\omega$ be a $\mathfrak{b}$-scale; that is, each b_α is increasing, $b_\alpha\le^* b_\beta$ for all $\alpha<\beta<\mathfrak{b}$, and B is unbounded.

As $|B|<\mathfrak{d}$, B is not dominating. Let $g\in\omega^\omega$ exemplify that. For each $\alpha<\mathfrak{b}$, let $x_\alpha=\{n:b_\alpha(n)\le g(n)\}$. Then $T=\{x_\alpha:\alpha<\mathfrak{b}\}$ is an unbounded tower: Clearly, $x_\beta\subseteq^* x_\alpha$ for $\alpha<\beta$. Assume that T is bounded, and let $f\in\omega^\omega$ exemplify that. For each α, writing $x_\alpha(n)$ for the n-th element of x_α:

$$b_\alpha(n)\le b_\alpha(x_\alpha(n))\le g(x_\alpha(n))\le g(f(n))$$

for all but finitely many n. Thus, $g\circ f$ shows that B is bounded, a contradiction. □

Theorem 2.3. *If there is an unbounded tower (of any cardinality), then there is a set of reals X of cardinality $\mathfrak{b}$ that satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

Theorem 2.3 follows from Propositions 2.4 and 2.5.

Proposition 2.4. *If there is an unbounded tower, then there is one of cardinality $\mathfrak{b}$.*

Proof. By Lemma 2.2, it remains to consider the case $\mathfrak{b}=\mathfrak{d}$. Let T be an unbounded tower of cardinality κ. Let $\{f_\alpha:\alpha<\mathfrak{b}\}\subseteq\omega^\omega$ be dominating. For each $\alpha<\mathfrak{b}$, pick $x_\alpha\in T$ which is not bounded by f_α. $\{x_\alpha:\alpha<\mathfrak{b}\}$ is unbounded, being unbounded in a dominating family. □

[2] Blass's survey [4] is a good reference for the definitions and details of the special cardinals mentioned in this paper.

Define a topology on $P(\omega)$ by identifying $P(\omega)$ with the Cantor space 2^ω, via characteristic functions. Scheepers's mentioned proof actually establishes the following result, to which we give an alternative proof.

Proposition 2.5 (essentially, Scheepers [16]). *For each unbounded tower T of cardinality $\mathfrak{b}$, $T \cup [\omega]^{<\omega}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

Proof. Let $T = \{x_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded tower of cardinality $\mathfrak{b}$. For each α, let $X_\alpha = \{x_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$. Let $\mathcal{U}_0, \mathcal{U}_1, \dots$ be point-cofinite open covers of $X_\mathfrak{b} = T \cup [\omega]^{<\omega}$. We may assume that each $\mathcal{U}_n$ is countable and that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ whenever $i \neq j$.

By the proof of Lemma 1.2 of [6], for each k there are distinct $U^k_0, U^k_1, \dots \in \mathcal{U}_k$, and an increasing sequence $m^k_0 < m^k_1 < \dots$, such that for each n and k,

$$\{x \subseteq \omega \ : \ x \cap (m^k_n, m^k_{n+1}) = \emptyset\} \subseteq U^k_n.$$

As T is unbounded, there is $\alpha < \mathfrak{b}$ such that for each k, $I_k = \{n : x_\alpha \cap (m^k_n, m^k_{n+1}) = \emptyset\}$ is infinite.

For each k, $\{U^k_n : n \in \omega\}$ is an infinite subset of $\mathcal{U}_k$, and thus a point-cofinite cover of X_α. As $|X_\alpha| < \mathfrak{b}$, there is $f \in \omega^\omega$ such that

$$\forall x \in X_\alpha \ \exists k_0 \ \forall k \geq k_0 \ \forall n > f(k) \ \ x \in U^k_n.$$

For each k, pick $n_k \in I_k$ such that $n_k > f(k)$.

We claim that $\{U^k_{n_k} : k \in \omega\}$ is a point-cofinite cover of $X_\mathfrak{b}$: If $x \in X_\alpha$, then $x \in U^k_{n_k}$ for all but finitely many k, because $n_k > f(k)$ for all k. If $x = x_\beta$, $\beta \geq \alpha$, then $x \subseteq^* x_\alpha$. For each large enough k, $m^k_{n_k}$ is large enough, so that $x \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) = \emptyset$, and thus $x \in U^k_{n_k}$. □

Remark 2.6. Zdomskyy points out that for the proof to go through, it suffices that $\{x_\alpha : \alpha < \mathfrak{b}\}$ is such that there is an unbounded $\{y_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^\omega$ such that for each α, x_α is a pseudointersection of $\{y_\beta : \beta < \alpha\}$. We do not know whether the assertion mentioned here is weaker than the existence of an unbounded tower.

We now turn to nontrivial examples of sets satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers. In general, $\mathsf{S}_1(\Gamma,\Gamma)$ is not preserved by taking finite powers [9], and we use a slightly stronger hypothesis in our construction.

Definition 2.7. Let $\mathfrak{b}_0$ be the additivity number of $\mathsf{S}_1(\Gamma,\Gamma)$, that is, the minimum cardinality of a family $\mathcal{F}$ of sets of reals, each satisfying $\mathsf{S}_1(\Gamma,\Gamma)$, such that the union of all members of $\mathcal{F}$ does not satisfy $\mathsf{S}_1(\Gamma,\Gamma)$.

$\mathfrak{t} \leq \mathfrak{h}$, and Scheepers proved that $\mathfrak{h} \leq \mathfrak{b}_0 \leq \mathfrak{b}$ [17]. It follows from Theorem 3.6 that, consistently, $\mathfrak{h} < \mathfrak{b}_0 = \mathfrak{b}$. It is open whether $\mathfrak{b}_0 = \mathfrak{b}$ is provable. If $\mathfrak{t} = \mathfrak{b}$ or $\mathfrak{h} = \mathfrak{b} < \mathfrak{d}$, then there is an unbounded tower of cardinality $\mathfrak{b}_0$.

Theorem 2.8. *For each unbounded tower T of cardinality $\mathfrak{b}_0$, all finite powers of $T \cup [\omega]^{<\omega}$ satisfy $\mathsf{S}_1(\Gamma,\Gamma)$.*

Proof. We say that $\mathcal{U}$ is an *ω-cover* of X if no member of $\mathcal{U}$ contains X as a subset, but each finite subset of X is contained in some member of $\mathcal{U}$. We need a multidimensional version of Lemma 1.2 of [6].

Lemma 2.9. *Assume that $[\omega]^{<\omega} \subseteq X \subseteq P(\omega)$, and let $e \in \omega$. For each open ω-cover $\mathcal{U}$ of X^e, there are $m_0 < m_1 < \dots$ and $U_0, U_1, \dots \in \mathcal{U}$, such that for all $x_0, \dots, x_{e-1} \subseteq \omega$, $(x_0, \dots, x_{e-1}) \in U_n$ whenever $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$.*

Proof. As $\mathcal{U}$ is an open ω-cover of X^e, there is an open ω-cover $\mathcal{V}$ of X such that $\{V^e : V \in \mathcal{V}\}$ refines $\mathcal{U}$ [9].

Let $m_0 = 0$. For each $n \geq 0$: Assume that $V_0, \dots, V_{n-1} \in \mathcal{V}$ are given, and $U_0, \dots, U_{n-1} \in \mathcal{U}$ are such that $V_i^e \subseteq U_i$ for all $i < n$. Fix a finite $F \subseteq X$ such that F^e is not contained in any of the sets $U_0, \dots, U_{n-1}$. As $\mathcal{V}$ is an ω-cover of X, there is $V_n \in \mathcal{V}$ such that $F \cup P(\{0, \dots, m_n\}) \subseteq V_n$. Take $U_n \in \mathcal{U}$ such that $V_n^e \subseteq U_n$. Then $U_n \notin \{U_0, \dots, U_{n-1}\}$. As V_n is open, for each $s \subseteq \{0, \dots, m_n\}$ there is k_s such that for each $x \in P(\omega)$ with $x \cap \{0, \dots, k_s - 1\} = s$, $x \in V_n$. Let $m_{n+1} = \max\{k_s : s \subseteq \{0, \dots, m_n\}\}$.

If $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$, then $(x_0, \dots, x_{e-1}) \in V_n^e \subseteq U_n$. □

The assumption in the theorem that there is an unbounded tower of cardinality $\mathfrak{b}_0$ implies that $\mathfrak{b}_0 = \mathfrak{b}$. The proof is by induction on the power e of $T \cup [\omega]^{<\omega}$. The case $e = 1$ follows from Theorem 2.5.

Let $\mathcal{U}_0, \mathcal{U}_1, \dots \in \Gamma((T \cup [\omega]^{<\omega})^e)$. We may assume that these covers are countable. As in the proof of Theorem 2.5 (this time using Lemma 2.9), there are for each k, $m_0^k < m_1^k < \dots$ and $U_0^k, U_1^k, \dots \in \mathcal{U}_k$ (so that $\{U_n^k : n \in \omega\} \in \Gamma((T \cup [\omega]^{<\omega})^e)$), such that for all $y_0, \dots, y_{e-1} \subseteq \omega$, $(y_0, \dots, y_{e-1}) \in U_n^k$ whenever $y_i \cap (m_n^k, m_{n+1}^k) = \emptyset$ for all $i < e$.

Let α_0 be such that $X_{\alpha_0}^e$ is not contained in any member of $\bigcup_n \mathcal{U}_n$. As T is unbounded, there is α such that $\alpha_0 \leq \alpha < \mathfrak{b}$, and for each k, $I_k = \{n : x_\alpha \cap (m_n^k, m_{n+1}^k) = \emptyset\}$ is infinite.

Let $Y = \{x_\beta : \beta \geq \alpha\}$. $(T \cup [\omega]^{<\omega})^e \setminus Y^e$ is a union of fewer than $\mathfrak{b}_0$ homeomorphic copies of $(T \cup [\omega]^{<\omega})^{e-1}$. By the induction hypothesis, $(T \cup [\omega]^{<\omega})^{e-1}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$, and therefore so does $(T \cup [\omega]^{<\omega})^e \setminus Y^e$. For each k, $\{U_n^k : n \in I_k\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$, and thus there are infinite $J_0 \subseteq I_0, J_1 \subseteq I_1, \dots$, such that $\{\bigcap_{n \in J_k} U_n^k : k \in \omega\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$.[3] For each k, pick $n_k \in J_k$ such that: $m_{n_k}^k > m_{n_{k-1}+1}^{k-1}$, $x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) = \emptyset$, and $U_{n_k}^k \notin \{U_{n_0}^0, \dots, U_{n_{k-1}}^{k-1}\}$.

$\{U_{n_k}^k : k \in \omega\} \in \Gamma(T \cup [\omega]^{<\omega})$: If $x \in (T \cup [\omega]^{<\omega})^e \setminus Y^e$, then $x \in U_{n_k}^k$ for all but finitely many k. If $x = (x_{\beta_0}, \dots, x_{\beta_{e-1}}) \in Y$, then $\beta_0, \dots, \beta_{e-1} \geq \alpha$, and thus $x_{\beta_0}, \dots, x_{\beta_{e-1}} \subseteq^* x_\alpha$. For each large enough k, $m_{n_k}^k$ is large enough, so that $x_{\beta_i} \cap (m_{n_k}^k, m_{n_k+1}^k) \subseteq x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) = \emptyset$ for all $i < e$, and thus $x \in U_{n_k}^k$. □

There is an additional way to obtain nontrivial $\mathsf{S}_1(\Gamma, \Gamma)$ sets: The hypothesis $\mathfrak{b} = \operatorname{cov}(\mathcal{N}) = \operatorname{cof}(\mathcal{N})$ provides $\mathfrak{b}$-Sierpiński sets, and $\mathfrak{b}$-Sierpiński sets satisfy $\mathsf{S}_1(\Gamma, \Gamma)$, even for *Borel* point-cofinite covers. Details are available in [18].

We record the following consequence of Theorem 2.3 for later use.

Corollary 2.10. *For each unbounded tower T of cardinality $\mathfrak{b}$, $T \cup [\omega]^{<\omega}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ for open covers, but not for Borel covers.*

Proof. The latter property is hereditary for subsets [18]. By a theorem of Hurewicz, a set of reals satisfies $\mathsf{U}_{\text{fin}}(\Gamma, \Gamma)$ if and only if each continuous image of X in ω^ω is bounded. It follows that the set $T \subseteq T \cup [\omega]^{<\omega}$ does not even satisfy $\mathsf{U}_{\text{fin}}(\Gamma, \Gamma)$. □

[3] Choosing infinitely many elements from each cover, instead of one, can be done by adding to the given sequence of covers all cofinite subsets of the given covers.

3. A consistency result

By the results of the previous section, we have the following.

Lemma 3.1. *Assume that every set of reals with property* $\mathsf{S}_1(\Gamma,\Gamma)$ *has cardinality* $< \mathfrak{b}$*, and* $\mathfrak{c} = \aleph_2$*. Then* $\aleph_1 = \mathfrak{t} = \mathrm{cov}(\mathcal{N}) < \mathfrak{b} = \aleph_2$*.*

Proof. As there is no unbounded tower, we have that $\mathfrak{t} < \mathfrak{b} = \mathfrak{d}$. As $\mathfrak{c} = \aleph_2$, $\aleph_1 = \mathfrak{t} < \mathfrak{b} = \aleph_2$. Since there are no $\mathfrak{b}$-Sierpiński sets and $\mathfrak{b} = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$, $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$. □

In Laver's model [11], $\aleph_1 = \mathfrak{t} = \mathrm{cov}(\mathcal{N}) < \mathfrak{b} = \aleph_2$. We will show that, indeed, $\mathsf{S}_1(\Gamma,\Gamma)$ is trivial there. Laver's model was constructed to realize Borel's Conjecture, asserting that "strong measure zero" is trivial. In some sense, $\mathsf{S}_1(\Gamma,\Gamma)$ is a dual of strong measure zero. For example, the canonical examples of $\mathsf{S}_1(\Gamma,\Gamma)$ sets are Sierpiński sets, a measure-theoretic object, whereas the canonical examples of strong measure zero sets are Luzin sets, a Baire category theoretic object. More about that can be seen in [18].

The main technical result of this paper is the following.

Theorem 3.2. *In the Laver model, if* $X \subseteq 2^\omega$ *has cardinality* $\mathfrak{b}$*, then there is a Borel map* $f : 2^\omega \to \omega^\omega$ *such that* $f[X]$ *is unbounded.*

Proof. The notation in this proof is as in Laver [11]. We will use the following slightly simplified version of Lemma 14 of [11].

Lemma 3.3 (Laver)**.** *Let* $\mathbb{P}_{\omega_2}$ *be the countable support iteration of Laver forcing,* $p \in \mathbb{P}_{\omega_2}$*, and let* $\mathring{a}$ *be a* $\mathbb{P}_{\omega_2}$*-name such that*

$$p \Vdash \mathring{a} \in 2^\omega.$$

Then there are a condition q *stronger than* p *and finite* $U_s \subseteq 2^\omega$ *for each* $s \in q(0)$ *extending the root of* $q(0)$ *such that for all such* s *and all* n*:*

$$q(0)_t \mathbin{\hat{}} q \restriction [1,\omega_2) \Vdash \text{“} \exists u \in \check{U}_s \;\; u \restriction n = \mathring{a} \restriction n \text{”}$$

for all but finitely many immediate successors t *of* s *in* $q(0)$*.*

Assume that $X \subseteq 2^\omega$ has no unbounded Borel image in $\mathcal{M}[G_{\omega_2}]$, i.e., Laver's model. For every code $u \in 2^\omega$ for a Borel function $f : 2^\omega \to \omega^\omega$ there exists $g \in \omega^\omega$ such that for every $x \in X$ we have that $f(x) \leq^* g$.

By a standard Löwenheim-Skolem argument (see Theorem 4.5 on page 281 of [3] or section 4 on page 580 of [12]), we may find $\alpha < \omega_2$ such that for every code $u \in \mathcal{M}[G_\alpha]$ there is an upper bound $g \in \mathcal{M}[G_\alpha]$. By the arguments employed by Laver [11, Lemmata 10 and 11], we may assume that $\mathcal{M}[G_\alpha]$ is the ground model $\mathcal{M}$.

Since the continuum hypothesis holds in $\mathcal{M}$ and $|X| = \mathfrak{b} = \aleph_2$, there are $p \in G_{\omega_2}$ and $\mathring{a}$ such that

$$p \Vdash \mathring{a} \in \mathring{X} \text{ and } \mathring{a} \notin \mathcal{M}.$$

Work in the ground model $\mathcal{M}$.

Let $q \leq p$ be as in Lemma 3.3. Define

$$Q = \{s \in q(0) \;:\; \mathrm{root}(q(0)) \subseteq s\}$$

and let U_s, $s \in Q$, be the finite sets from the lemma. Let $U = \bigcup_{s \in Q} U_s$. Define a Borel map $f : 2^\omega \to \omega^Q$ so that for every $x \in 2^\omega \setminus U$ and for each $s \in Q$: If

$f(x)(s) = n$, then $x \restriction n \neq u \restriction n$ for each $u \in U_s$. For $x \in U$, $f(x)$ may be arbitrary. There must be a $g \in \omega^Q \cap \mathcal{M}$ and $r \leq q$ such that

$$r \Vdash f(\mathring{a}) \leq^* \check{g}.$$

Since p forced that a is not in the ground model, it cannot be that a is in U. We may extend $r(0)$ if necessary so that if $s = \mathrm{root}(r(0))$, then

$$r \Vdash f(\mathring{a})(s) \leq \check{g}(s).$$

But this is a contradiction to Lemma 3.3, since for all but finitely many $t \in r(0)$ which are immediate extensions of s:

$$r(0)_t \,\hat{}\, q \restriction [1, \omega_2) \Vdash f(\mathring{a})(s) > \check{g}(s). \qquad \square$$

In [20], Tsaban and Zdomskyy prove that $\mathsf{S}_1(\Gamma, \Gamma)$ for Borel covers is equivalent to the Kočinac property $\mathsf{S}_{\mathrm{cof}}(\Gamma, \Gamma)$ [10], asserting that for all $\mathcal{U}_0, \mathcal{U}_1, \dots \in \Gamma$, there are cofinite subsets $\mathcal{V}_0 \subseteq \mathcal{U}_0, \mathcal{V}_1 \subseteq \mathcal{U}_1, \dots$ such that $\bigcup_n \mathcal{V}_n \in \Gamma$. The main result of [5] can be reformulated as follows.

Theorem 3.4 (Dow [5])**.** *In Laver's model,* $\mathsf{S}_1(\Gamma, \Gamma)$ *implies* $\mathsf{S}_{\mathrm{cof}}(\Gamma, \Gamma)$.

For the reader's convenience, we give Dow's proof, adapted to the present notation.

Proof. A family $\mathcal{H} \subseteq [\omega]^\omega$ is *ω-splitting* if for each countable $\mathcal{A} \subseteq [\omega]^\omega$, there is $H \in \mathcal{H}$ which splits each element of $\mathcal{A}$, i.e.,

$$|A \cap H| = |A \setminus H| = \omega \text{ for all } A \in \mathcal{A}.$$

The main technical result in [5] is the following.

Lemma 3.5 (Dow)**.** *In Laver's model, each ω-splitting family contains an ω-splitting family of cardinality $< \mathfrak{b}$.*

Assume that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. Let $\mathcal{U}_0, \mathcal{U}_1, \dots$ be open point-cofinite countable covers of X. We may assume[4] that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ whenever $i \neq j$. Put $\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$. We identify $\mathcal{U}$ with ω, its cardinality.

Define $\mathcal{H} \subseteq [\mathcal{U}]^\omega$ as follows. For $H \in [\mathcal{U}]^\omega$, put $H \in \mathcal{H}$ if and only if there exists $\mathcal{V} \in [\mathcal{U}]^\omega$, a point-cofinite cover of X, such that $H \cap \mathcal{U}_n \subseteq^* \mathcal{V}$ for all n. We claim that $\mathcal{H}$ is an ω-splitting family. As $\mathcal{H}$ is closed under taking infinite subsets, it suffices to show that it is *ω-hitting*; i.e., for any countable $\mathcal{A} \subseteq [\mathcal{U}]^\omega$ there exists $H \in \mathcal{H}$ which intersects each $A \in \mathcal{A}$. (It is enough to intersect each $A \in \mathcal{A}$, since we may assume that $\mathcal{A}$ is closed under taking cofinite subsets.)

Let $\mathcal{A} \subseteq [\mathcal{U}]^\omega$ be countable. For each n, choose sets $\mathcal{U}_{n,m} \in [\mathcal{U}_n]^\omega$, $m \in \omega$, such that for each $A \in \mathcal{A}$, if $A \cap \mathcal{U}_n$ is infinite, then $\mathcal{U}_{n,m} \subseteq A$ for some m. Apply the $\mathsf{S}_1(\Gamma, \Gamma)$ to the family $\{\mathcal{U}_{n,m} : n, m \in \omega\}$ to obtain a point-cofinite $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \cap \mathcal{U}_{n,m}$ is nonempty for all n, m.

Next, choose finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that for each $A \in \mathcal{A}$ with $A \cap \mathcal{U}_n$ finite for all n, then $A \subseteq^* \bigcup_n \mathcal{F}_n$. Take $H = \mathcal{V} \cup \bigcup_n \mathcal{F}_n$. Then H is in $\mathcal{H}$ and meets each $A \in \mathcal{A}$. This shows that $\mathcal{H}$ is an ω-splitting family.

By Lemma 3.5, there is an ω-splitting $\mathcal{H}' \subseteq \mathcal{H}$ of cardinality $< \mathfrak{b}$. For each $H \in \mathcal{H}'$, let $\mathcal{V}_H$ witness that H is in $\mathcal{H}$; i.e., $\mathcal{V}_H \subseteq \mathcal{U}$ is a point-cofinite cover of X and $H \cap \mathcal{U}_n \subseteq^* \mathcal{V}_H$ for all n.

[4]To see why, replace each $\mathcal{U}_n$ by $\mathcal{U}_n \setminus \bigcup_{i<n} \mathcal{U}_i$ and discard the finite ones. It suffices to show that $\mathsf{S}_{\mathrm{cof}}(\Gamma, \Gamma)$ applies to those that are left.

By the definition of $\mathfrak{b}$, we may find finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that for each $H \in \mathcal{H}'$,

$$H \cap \mathcal{U}_n \subseteq \mathcal{V}_H \cup \mathcal{F}_n$$

for all but finitely many n. We claim that $\mathcal{W} = \bigcup_n \mathcal{U}_n \setminus \mathcal{F}_n$ is point-cofinite. Suppose it is not. Then there is $x \in X$ such that for infinitely many n, there is $U_n \in \mathcal{U}_n \setminus \mathcal{F}_n$ with $x \notin U_n$. Let $H \in \mathcal{H}'$ contain infinitely many of these U_n. By the above inclusion, all but finitely many of these U_n are in $\mathcal{V}_H$. This contradicts the fact that $\mathcal{V}_H$ is point-cofinite. □

We therefore have the following.

Theorem 3.6. *In Laver's model, each set of reals X satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ has cardinality less than $\mathfrak{b}$.*

Proof. By Dow's Theorem, $\mathsf{S}_1(\Gamma,\Gamma)$ implies $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$, which in turn implies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers [20]. The latter property is equivalent to having all Borel images in ω^ω bounded [18]. Apply Theorem 3.2. □

Thus, it is consistent that strong measure zero and $\mathsf{S}_1(\Gamma,\Gamma)$ are both trivial.

The proof of Dow's Theorem 3.4 becomes more natural after replacing, in Lemma 3.5 "ω-splitting" by "ω-hitting". This is possible, due to the following fact (cf. Remark 4 of [5]).

Proposition 3.7. *For each infinite cardinal κ, the following are equivalent:*

(1) *Each ω-splitting family contains an ω-splitting family of cardinality $< \kappa$.*
(2) *Each ω-hitting family contains an ω-hitting family of cardinality $< \kappa$.*

Proof. ($1 \Rightarrow 2$) Suppose $\mathcal{A}$ is an ω-hitting family. Let $\mathcal{B} = \bigcup_{A \in \mathcal{A}} [A]^\omega$. Then $\mathcal{B}$ is ω-splitting. By (1) there exists $\mathcal{C} \subseteq \mathcal{B}$ of size $< \kappa$ which is ω-splitting. Choose $\mathcal{D} \subseteq \mathcal{A}$ of size $< \kappa$ such that for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ with $C \subseteq D$. Then $\mathcal{D}$ is ω-hitting.

($2 \Rightarrow 1$) Suppose $\mathcal{A}$ is an ω-splitting family. For each $A \subseteq \omega$ define

$$A^* = \{2n : n \in A\} \cup \{2n+1 : n \in \overline{A}\}.$$

Then the family $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is ω-hitting. To see this, suppose that $\mathcal{B}$ is countable. Without loss we may assume that $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, where each element of $\mathcal{B}_0$ is a subset of the evens and each element of $\mathcal{B}_1$ is a subset of the odds. For $B \in \mathcal{B}_0$ let $C_B = \{n : 2n \in B\}$ and for $B \in \mathcal{B}_1$ let $C_B = \{n : 2n+1 \in B\}$. Now put

$$\mathcal{C} = \{C_B : B \in \mathcal{B}\}.$$

Since $\mathcal{A}$ is ω-splitting there is $A \in \mathcal{A}$ which splits $\mathcal{C}$. If $B \in B_0$, then $A \cap C_B$ infinite implies $B \cap A^*$ infinite. If $B \in B_1$, then $\overline{A} \cap C_B$ infinite implies $B \cap A^*$ infinite.

By (2) there exists $\mathcal{A}_0 \subseteq \mathcal{A}$ of cardinality $< \kappa$ such that $\mathcal{A}_0^*$ is ω-hitting. We claim that $\mathcal{A}_0$ is ω-splitting. Given any $B \subseteq \omega$ let $B' = \{2n : n \in B\}$ and let $B'' = \{2n+1 : n \in B\}$. Given $\mathcal{B} \subseteq [\omega]^\omega$ countable, there exists $A \in \mathcal{A}_0$ such that A^* hits each B' and B'' for $B \in \mathcal{B}$. But this implies that A splits B. □

4. Applications to Arhangel'skiĭ's α_i spaces

Let Y be a general (not necessarily metrizable) topological space. We say that a countably infinite set $A \subseteq Y$ converges to a point $y \in Y$ if each (equivalently, some) bijective enumeration of A converges to y. The following concepts are due to Arhangel'skiĭ. Y is an α_1 *space* if for each $y \in Y$ and each sequence $A_0, A_1, \ldots$ of countably infinite sets, each converging to y, there are cofinite $B_0 \subseteq A_0, B_1 \subseteq A_1, \ldots$, such that $\bigcup_n B_n$ converges to y. Replacing "cofinite" by "singletons" (or equivalently, by "infinite"), we obtain the definition of an α_2 *space*.

We first consider countable spaces.

Definition 4.1. Let X be a set of reals, and let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be countable point-cofinite covers of X. For each n, enumerate bijectively $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. We associate to X a (new) topology τ on the fan $S_\omega = \omega \times \omega \cup \{\infty\}$ as follows: ∞ is the only nonisolated point of S_ω, and a neighborhood base at ∞ is given by the sets

$$[\infty]_F = \{(n,m) : F \subseteq U^n_m\}$$

for each finite $F \subseteq X$.

Lemma 4.2. *In the notation of Definition* 4.1*: A converges to ∞ in τ if and only if $\mathcal{U}(A) = \{U^n_m : (n,m) \in A\}$ is a point-cofinite cover of X.* □

Assume that there is an unbounded tower. By Corollary 2.10, there is a set of reals X satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ but not $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be countable open point-cofinite covers of X witnessing the failure of $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$. Then, by Lemma 4.2, (S_ω, τ) is α_2 but not α_1. In particular, we reproduce the following.

Corollary 4.3 (Nyikos [13])**.** *If there is an unbounded tower of cardinality $\mathfrak{b}$, then there is a countable α_2 space which is not an α_1 space.* □

Recall that by Proposition 2.4, it suffices to assume in Corollary 4.3 the existence of any unbounded tower.

Next, we consider spaces of continuous functions. Consider $C(X)$, the family of continuous real-valued functions, as a subspace of the Tychonoff product $\mathbb{R}^X$, i.e., with the topology of pointwise convergence. Sakai [14] proved that X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for *clopen* covers if and only if $C(X)$ is an α_2 space. The main result of [20] is that $C(X)$ is α_1 if and only if X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers (equivalently, each Borel image of X in ω^ω is bounded).

The *Scheepers Conjecture* is that for subsets of $\mathbb{R} \setminus \mathbb{Q}$, $\mathsf{S}_1(\Gamma,\Gamma)$ for clopen covers implies $\mathsf{S}_1(\Gamma,\Gamma)$ for open covers. Dow [5] proved that in Laver's model, every α_2 space is α_1. By Theorem 3.2, we can add the last item in the following list.

Corollary 4.4. *In Laver's model, the following are equivalent for sets of reals X:*

1. *$C(X)$ is an α_2 space;*
2. *$C(X)$ is an α_1 space;*
3. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for clopen covers;*
4. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for open covers;*
5. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers;*
6. *$|X| < \mathfrak{b}$.* □

On the other hand, Corollary 2.10 implies the following.

Corollary 4.5. *If there is an unbounded tower, then there is a set of reals X such that $C(X)$ is α_2 but not α_1.* □

Essentially, Corollary 4.3 is a special case of Corollary 2.10, whereas Corollary 4.5 is equivalent to Corollary 2.10.

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