1 The history of set theory

1.1 The concept of infinity

The idea of infinity had been the subject of deep thought from the time of the Greeks. Zeno of Elea, in around 450 BC, with his problems on the infinite, made an early major contribution. By the Middle Ages discussion of the infinite had led to comparison of infinite sets. For example Albert of Saxony, in `Questiones subtilissime in libros de celo et mundi`, proves that a beam of infinite length has the same volume as 3-space. He proves this by sawing the beam into imaginary pieces which he then assembles into successive concentric shells which fill space.

1.2 Infinite sets

Bolzano was a philosopher and mathematician of great depth of thought. In 1847 he considered sets with the following definition

> *an embodiment of the idea or concept which we conceive when we regard the arrangement of its parts as a matter of indifference.*

Bolzano defended the concept of an infinite set. At this time many believed that infinite sets could not exist. Bolzano gave examples to show that, unlike for finite sets, the elements of an infinite set could be put in 1-1 correspondence with elements of one of its proper subsets. This idea eventually came to be used in the definition of a finite set.

It was with Cantor's work however that set theory came to be put on a proper mathematical basis.

1.3 Cantor - The pioneer

in 1872 Cantor met Richard Dedekind and a friendship grew up that was to last for many years. it is clear that Dedekind's deep abstract logical way of thinking was a major influence on Cantor as his ideas developed.

Cantor moved from number theory to papers on trigonometric series. These papers contain Cantor's first ideas on set theory and also important results on irrational numbers. Dedekind was working independently on irrational numbers and Dedekind published `Continuity and irrational numbers`.

1.3.1 Sizes of sets

In 1874 Cantor published an article in which he considers at least two different kinds of infinity. Before this orders of infinity did not exist but all infinite collections were considered 'the same size'. However Cantor examines the set of algebraic real numbers, that is the set of all real roots of equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \tag{1}$$

where a_i is an integer.Cantor proved that the algebraic real numbers are in one-one correspondence with the natural numbers.

In the same paper Cantor shows that the real numbers cannot be put into one-one correspondence with the natural numbers.Cantor now remarks that this proves a theorem due to Liouville, namely that there are infinitely many transcendental (i.e. not algebraic) numbers in each interval.

1.3.2 The controversy

A follow-up paper was submitted by Cantor to Crelle's Journal in 1878 but already set theory was becoming the centre of controversy. Kronecker, who was on the editorial staff of Crelle's Journal, was unhappy about the revolutionary new ideas contained in Cantor's paper. Cantor was tempted to withdraw the paper but Dedekind persuaded Cantor not to withdraw it and Weierstrass supported publication.

In this article Cantor introduces the idea of equivalence of sets and says two sets are equivalent or have the same **power** if they can be put in 1-1 correspondence. The word '**power**' Cantor took from Steiner. He proves that the rational numbers have the smallest infinite power and also shows that $\Re^n$ has the same power as $\Re$. He shows further that countably many copies of $\Re$ still has the same power as $\Re$. At this stage Cantor does not use the word **countable**, but he was to introduce the word in a paper of 1883.

Kronecker's criticism was built on the fact that he believed only in constructive mathematics. He only accepted mathematical objects that could be constructed finitely from the intuitively given set of natural numbers. When Lindemann proved that π is transcendental in 1882 Kronecker said

> *Of what use is your beautiful investigation of π. Why study such problems when irrational numbers do not exist.*

Certainly Cantor's array of different infinities were impossible under this way of thinking.

Cantor however continued with his work.He published a six part treatise on set theory from the years 1879 to 1884. This work appears in `Mathematische Annalen` His fifth work in the six part treatise was published in 1883 and discusses **well-ordered** sets. **Ordinal numbers** are introduced as the order types of well-ordered sets. Multiplication and addition of **transfinite numbers** are also defined in this work (although Cantor was to give a fuller account of transfinite arithmetic in later work).

Cantor takes quite a portion of this article justifying his work. He claimes that mathematics is quite free and any concepts may be introduced subject only to the condition that they are free of contradiction and defined in terms of previously accepted concepts. He also cites many previous authors who had given opinions on the concept of infinity including Aristotle Descartes, Berkeley, Leibniz and Bolzano.

In 1897 the first International Congress of Mathematicians was held in Zurich and at that conference Cantor's work was held in the highest esteem being praised by many including Hurwitz and Hadamard.

1.3.3 Without controversy

In 1885 Cantor continued to extend his theory of **cardinal numbers** and of order types. He extended his theory of order types so that now his previously defined ordinal numbers became a special case. In 1895 and 1897 Cantor published his final double treatise on sets theory. It contains an introduction that looks like a modern book on set theory, defining set, subset, etc. Cantor proves that if A and B are sets with A equivalent to a subset of B and B equivalent to a subset of A then A and B are equivalent.

1.3.4 An internal enemy

In 1897 the first published **paradox** appeared, published by Cesare Burali-Forti. It basically revolves round the set of all ordinal numbers. The ordinal number of the set of all ordinals must be an ordinal and this leads to a contradiction.

In 1899 Cantor discovered another paradox which arises from the set of all sets. What is the cardinal number of the set of all sets? Clearly it must be the greatest possible cardinal yet the cardinal of the set of all subsets of a set always has a greater cardinal than the set itself. It began to look as if the criticism of Kronecker might be at least partially right since extension of the set concept too far seemed to be producing the paradoxes. The 'ultimate' paradox was found by Russell in 1902. It simplify defined a set

$$A = \{X | X \notin X\}. \tag{2}$$

Russell then asked :- *Is A an element of A?* Both the assumption that A is a member of A and A is not a member of A lead to a contradiction. The set construction itself appears to give a paradox.

1.4 There might be a solution...

Did the paradoxes come from the '**Axiom of choice**'? In 1890 peano note that Cantor was using the 'Axiom of choice' without feeling that it was necessary to single it out for any special treatment.

Actually the first to formally introduce the axiom of choice was Zermelo when he proved, in 1904, that every set can be well-ordered. This theorem had been conjectured by Cantor. Émile Borel pointed out that the Axiom of Choice is in fact equivalent to Zermelo's Theorem.

Gödel showed, in 1940, that the Axiom of Choice cannot be disproved using the other axioms of set theory. It was not until 1963 that Paul Cohen proved that the Axiom of Choice is independent of the other axioms of set theory.

Russell's paradox had *undermined the whole of mathematics* in Frege's words. Russell, trying to repair the damage, made an attempt to put mathematics back onto an logical basis in his major work `Principia Mathematica` written with Whitehead. This work attempts to reduce the foundations of mathematics to logic and was extremely influential. However the method of avoiding the paradoxes by introducing a 'theory of types' made it impossible to say that a class was or was not a member of itself. It did not seem a very satisfactory way around the problems and others sought different ways.

1.5 Congratulations! you have a new theory!

Zermelo in 1908 was the first to attempt an axiomatisation of set theory. Many other mathematicians attempted to axiomatise set theory. Fraenkel, von Neumann, Bernays and Gödel are all important figures in this development. Gödel showed the limitations of any axiomatic theory and the aims of many mathematicians such as Frege and Hilbert could never be achieved.

`This summery  based on an article of:J J O'Connor and E F Robertson.  taken from`
`http://www-history.mcs.st-andrews.ac.uk/history/HistTopics/Beginning_of_set_theory.html`
`or http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Beginning_of_set_theory.html`

Note:the bolded phrases will be discussed later in the course.

2 The idea behind the axiomatic Set Theory

Cantor started proving his theories formally with the definition of the axiomatic set theory. Why the axioms?

Russel's paradox
Let's define the set: $V = \{A : A \notin A\}$
The question is whether $V \in V$.

Let's assume that $V \notin V$ but then, by the definition of set V, $V \in V$, and we reach a contradiction.
On the other hand if $V \in V$ then by the definition of the elements in V, $V \notin V$, and we reach a contradiction again.
Our basic logic, that every element is always either in a set or not in a set was enough to lead us to this contradiction.

This paradox puts the naive mathematics in an embarrassing light. Cantor's "dream" was to build a simple and clean of contradictions axiomatic system, from which it would be possible to get the whole mathematics. The most natural place to start with was set theory.

To avoid contradictions mathematicians chose the formal attitude to the subject: we don't know exactly what sets are, but we do know what conditions they should satisfy. The basic qualities of sets will be defined by **axioms**. We can think of the sets as of "soldiers" in a game, and of axioms we can think as of "game rules" which define which kind of soldiers there are and what kind of soldiers we can get when we have certain soldiers ("supposing you have soldiers of type x and a soldier of type y, then additionally you will get type z soldier").

We will present variation of Zermelo and Fraenkhel's approach, called ZFC (ZF for their initials, and C for the "Choice axiom").

The modern set theory deals only with sets exploration. Therefore, in set theory when we say "for all x" we mean "for all set x". One of the changes regarding the naive set theory, is that now the sets elements are sets themselves. The immediate question is: what about famous elements like $1, 2, 3, \ldots$,which are not "sets"? Apparently they can also be defined as sets, and as a matter of fact everything in mathematics could be defined using sets.

3 The set theory Language

The use of the daily language when dealing with the set theory can also lead to paradoxes.

Exercise: a. Explain why the set of sentences containing less than 100 words is a final set?
b. Prove that a natural number n exists, which cannot be described in less than 100 words.
c. Let n be the smallest number, that cannot be described in less than 100 words. Show that the sentence you have just read describes n in less than 100 words. What is the contradiction?

In order to avoid contradictions of this kind, while exploring the set theory we will use accurate and single valued language. The set theory axioms, theorems, and even the profs can be formulated using this language. In practise we will use formulations in English language, but we will do so only when it is clear that we can also formulate them using the set theory language.

Like every language, our language has letters, which are combined to create sentences. The letters of our language would be: $\in$ (*belonging*), $=$ (*equality*), $(,)$ (*brackets indicate how to read the sentences*), logical connectors $\neg$ (*not*), $\wedge$ (*and*), $\vee$ (*or*), $\rightarrow$ (*consequently*), $\leftrightarrow$ (*equivalence*), the kamatim $\forall$ (for all), $\exists$ (exists), and the words that will indicate sets $A, B, C, \ldots, a, b, c, \ldots, \alpha, \beta, \gamma, \ldots, etc.$ Letters that indicate sets will be also called **variables**.

Not every sequence of words in a language can be given a meaning.

Question: Which of the following sequences has meaning?
a. $\forall x \forall y(x \in y)$.
b. $x \neg \rightarrow \wedge z \forall$.
c. $\forall x(x = y)$.

A legitimate sentence in set theory language will be called a set theory **formula** (*formula indicates a quality that sets possess or don't*).
The signs $a \in b$ or $a = b$, when a, b indicate sets, are called **basic formula**.
In a more general way, φ is a set **quality** if one of the following options occurs:
1. φ is a basic formula, that is either $a \in b$ or $a = b$.
2. There are formulas χ, ψ so that φ is one of the following options:
$(\neg\psi)$, $(\varphi \wedge \chi)$, $(\varphi \vee \chi)$, $(\varphi \rightarrow \chi)$, $(\varphi \leftrightarrow \chi)$.
3. There is a formula ψ so that φ is $(\exists x \psi)$ or $(\forall x \psi)$, x indicates set.

According to this a set theory formula is a sequence of letters received from basic formulas by operating rules (2) and (3) several times on the basic formulas (*and on what we get from them*).

Unbound variables. If formula φ has an unbound variable (*that is not "cap-*

tured" by a kamat), then $\varphi(a)$ indicates the instantiation of a instead of the unbound variable. For example if φ is the quality $(\forall y(x = y))$, then the variable x is unbound in φ and therefore $\varphi(a)$ is the formula $\forall y(a = y)$. Formally the term **unbound variable** is defined in induction:
1. The unbound variables in formulas $x \in y$ and $x = y$ are x and y.
2. The unbound variables in formulas of one of the following structures:

$$(\varphi \wedge \chi),\ (\varphi \vee \chi),\ (\varphi \rightarrow \chi),\ (\varphi \leftrightarrow \chi).$$

are the variables that are unbound in ψ or in χ.
3. The unbound variables in the formula $(\neg\psi)$ are those that are unbound in the formula ψ.
4. The unbound variables in the formulas $(\exists x\psi)$ and $(\forall x\psi)$ are those that are unbound in ψ, except x.

4 Axioms

This section will present set theory axioms, exercises and discussions in this matter.

4.1 Existence and Extensionality

The first axioms says that a set exists. For the matter of simplicity, existence of a set with no elements will be sufficient enough for us.

Axiom of Existence: Exists set A, so that there is no x for which $x \in A$.

We can think of letters $A, B, C, \ldots, a, b, c, \ldots$, which represent sets as of names of sets. According to this one set may have two names (*for example A and B*). The next axiom allows us to recognize this situation by looking at the set elements. The axiom says that every set is determined by it's elements.

Axiom of Extensionality: If A, B are sets, so that for all a $a \in A \leftrightarrow a \in B$, then $A = B$.

4.1.1 exercise

Write the Extensionality axiom as a formula.

For two sets A, B we'll say that A is **partial to** B, and we'll write $A \subseteq B$, if for all $a \in A$ there is also $a \in B$. If $A \subseteq B$ and $A \neq B$, we'll write $A \subset B$.

4.1.2 exercise

Prove that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Set which only elements are $a, b, c, \ldots$ will be marked $\{a, b, c, \ldots\}$

4.1.3 exercise

Prove:
a. If $A \neq B$, then $\exists a \in A$ so that $a \notin B$, or $\exists b \in B$ so that $b \notin A$.
b. If $a \notin A \leftrightarrow a \notin B$, then $A = B$.

4.1.4 exercise

Prove: If $\exists a \in A$ so that $a \notin B$, then $A \neq B$.

4.1.5 exercise

Prove: If $\forall a \; a \notin A$, and $\forall b \; b \notin B$, then $A = B$.

The last exercise showed that there is only one set with no elements. We'll sign this set as $\emptyset$, and it will be called **the empty set**.

4.1.6 exercise

Prove that for every set A, $A \subseteq A$, and $\emptyset \subseteq A$.

4.2 Regularity

As was said in the above, set's elements are sets. The next axiom is meant to prevent strange phenomenons.

Axiom of Regularity: If $A \neq \emptyset$ then $\exists a \in A$ so that if $b \in a$, then $b \notin A$. (*Given any nonempty set, there is an element whose intersection with the set is empty.*)

4.2.1 exercise

Write the suitable formula for the fundamentality axiom.

4.2.2 exercise

Assume that the following sets exist, and show that they satisfy the fundamentality axiom:
a. $\emptyset$.
b. $\{\{\emptyset\}\}$.
c. $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

The next exercise is an example for a strange phenomenon that the axiom prevents.

4.2.3 exercise

Can there be a set $\{A_1, A_2, A_3, \ldots\}$ so that for every $i = 1, 2, 3, \ldots A_{i+1} \in A_i$?

4.3 Separation

The following axiom allows to define a subset of a given set by rule that was determined by a formula.

Axiom of Separation: For every formula $\varphi(x, y)$ (*which doesn't have any unbound variables except for may be x and y*), we'll define the next axiom: for every set A, and a set p, exists the set $B = \{a \in A : \varphi(a, p)\}$.

4.3.1 exercise

Dany thinks that he have found a contradiction in the axioms. Help Dany to find the mistake in the following claim:
Suppose $A \neq \emptyset$ is a set, and we'll define a subset B of the set A by $B = \{a \in A : a \notin B\}$ (*Danny used the axiom of Separation when the formula ϕ is $\neg(a \in B)$*). We'll take some $a \in A$. If $a \in B$, then according to definition of B, $a \notin B$. Therefore $B = \emptyset$. From the other hand, for all $a \in A, a \notin B$ (*since B is empty*), but from the definition of B, it means that $a \in B$. Eventually we have that for all $a \in A, a \in B$. Therefore $A \subseteq B$. Since $B = \emptyset$, there is also $A = \emptyset$, as a contradiction for the assumption that $A \neq \emptyset$. Therefore all the sets are empty, meaning that there is only one set - the empty set.

4.3.2 exercise

Let φ be a formula and let A be a set. Prove that the following sets exist: $A_1 = \{a \in A : \varphi(a)\}, A_2 = \{a \in A : \neg\varphi(a)\}$. (*Verbally, axiom of Separation allows to "separate" A to two sets according to rule φ.*)

The next exercise defines some basic operations on sets.

4.3.3 exercise

Let A, B be any sets. Prove that the following sets exist:
a. $A \cap B := \{a : a \in A \text{ and } a \in B\}$.
b. $A \setminus B := \{a : a \in A \text{ and } a \notin B\}$.

4.3.4 exercise

Let F be a not empty set. Prove that the set $\cap F = \{x : \forall A \in F (x \in A)\}$ exists. [*hint: start with any set $X \in F$.*]

While in the naive set theory have been paradoxes, in the set theory that we are developing now paradoxes show that certain "collections" are not sets: by negation we assume that they are sets, and eventually come to a contradiction. The contradiction is to assumption that they are sets, therefore they do not exist as sets.

4.3.5 exercise (universal set)

a. Prove that there is no set U, so that for every set A, $A \in U$. Do it without using the axiom of Regularity. [*guidance: let's assume by negation that such set U exists. Show that the set $B = \{a \in U : a \notin a\}$ exists. Use Russel's paradox.*]
b. Explain why the definition of $\cap F$ has meaning only when F is not empty.

Now we have enough axioms to prove that the pathological case $A \in A$ can not take place.

4.3.6 exercise

Can there be a set A that satisfies $A \in A$? [*hint: by negation, prove the existence of the set $\{A\}$ (separation). And show that it contradicts the Axiom of Regularity.*]

Pay attention, that till now we know that exists only one set - the empty set $\emptyset$. In order to build more sets we'll be needing more axioms.

4.4 Pairing

Axiom of Pairing: If a, b are sets, then so is $A = \{a, b\}$.

Now we can build sets with one element, and with two elements.

4.4.1 exercise

a. Prove the existence of the following sets: $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}$.
b. Explain (do not prove) why you can't build a set with three elements?

For every a, b, we'll define the **ordered pair**r $< a, b >$ to be the set $\{\{a, \}, \{a, b\}\}$.

4.4.2 exercise

Show that the set $\{\{b\}, \{b, \{\{b\}\}\}\}$ is an ordered pair (*what are the elements of this ordered pair*)?

4.4.3 exercise

a. Prove that for every a, b exists the ordered pair $< a, b >$.
b. Prove that if $< a, b >=< c, d >$, then $a = c$ and $b = d$.

4.4.4 exercise

Dany thinks that the definition of the ordered pair is too complicated. He suggests to define $< a, b >= \{a, \{b\}\}$. Show Dany, using an example, that according to his definition possible that $< a, b >=< c, d >$ although $a \neq c, b \neq d$.

Here is another pathological case, that cannot exist according to axioms we have seen so far.

4.4.5 exercise

Can there be sets A, B, so that $A \in B \in A$? [*hint: prove the existence of the set* $\{A, B\}$.]

4.5 Union

The next axiom makes it possible to unify sets.

Axiom of Union: Given any set of sets S, there is a set $\cup S = \{x : \exists A \in S(x \in A)\}$ which is the union of all the sets in S.

4.5.1 exercise

Prove that for every two sets A, B exist the sets:
a. $A \cup B := \{a : a \in A \text{ or } b \in B\}$.
b. $A \triangle B := (A \backslash B) \cup (B \backslash A)$.

Now we can build sets with three, four and more elements.

4.5.2 exercise

Build a set with 3 elements, and a set with 4 elements. [*hint: for example,* $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.]

4.6 Functions and Replacement

Given any set A. We'll say that formula φ is a **correlation rule** upon A if for every $x \in A$ exists only one y satisfying the quality $\varphi(x, y)$.

Axiom of Replacement: For every formula $\varphi(x, y, z, w,)$ (*which doesn't have any unbound variables except maybe for* $x, y, z, w,$) defined the next axiom: for every given set A and a set p, if $\varphi(x, y, A, p,)$ is a correlation rule upon A, then the set $B = \{y : \exists x \in A\varphi(x, y, A, p,)\}$ exists.

4.6.1 exercise

Given any sets A, B.
a. Prove that for every $b \in B$, exists the set $A \times \{b\} := \{< a, b >: a \in A\}$. [*hint: for every* $a \in A$ *exists a unique* z *that satisfies* $z =< a, b >$.]

b. Prove the existence of the cartesian multiplication $A \times B := \{< a, b >: a \in A, b \in B\}$. [*hint:* $A \times B := \cup\{A \times \{b\} : b \in B\}$.]

Given sets A, B and $f \subseteq A \times B$ there is a **function** $f : A \to B$ if: $\forall a \in A$ there is a **unique** $b \in B$ so that $< a, b >\in f$ (*in such case we'll sign it as* $f(a) = b$).

4.6.2 exercise

What is the difference between function and a correlation rule? [*hint: function is a set, and correlation rule is a ...*]

If $f : A \to B$ is a function, then the set A is a the **domain** of f ($A = dom(f)$), and the set B is the **range** of f ($B = ran(f)$).
Although function and correlation rule are two different terms, every function determines a correlation rule, and every correlation rule defines a function with similar qualities. The next exercise defines this precisely.

4.6.3 exercise (the relationship between functions and correlation rules)

a. Given the function $f : A \to B$, and φ the formula $f(a) = b$. Show that φ is a correlation rule upon A.
b. Given φ a correlation rule upon A. Prove that exists a function f with $dom(f) = A$, so that for every $a \in A$, $f(a) = b \leftrightarrow \varphi(a, b)$.

The **image** of f is the set $im(f) := \{f(a) : a \in A\}$.

4.6.4 exercise

Given the function $f : A \to B$.
a. Prove that the image $im(f)$ exists as a set.
b. (*generation of (a)*) Prove that for every $\tilde{A} \subseteq A$, exists the set $f[\tilde{A}] := \{f(a) : a \in \tilde{A}\}$.

4.6.5 exercise

Prove that if $f : A \to B$ is a function, then for every C such as $im(f) \subseteq C \subseteq B$, $f : A \to C$ is a function.

If for every $b \in im(f)$ there is only one a so that $f(a) = b$, f will be called an **injection**.
f is a **surjection** if $im(f) = B$.
f is a **bijection** if f is an injection and a surjection.

4.6.6 exercise

Given a function $f : A \to B$, $\tilde{A} \subseteq A$. Prove that exists a unique function $g : \tilde{A} \to B$, so that for every $a \in \tilde{A}$ $g(a) = f(a)$. [*hint:* $g = f \cap (\tilde{A} \times B)$.]

The function in the last exercise will be called **tzimtzum** of f into the set $\tilde{A}$, and will be signed as $f|_{\tilde{A}}$.

4.6.7 exercise

Given any sets A, B, prove the existence of the next functions:
a. $\pi_1 : A \times B \to A$ such that for every $< a, b > \in A \times B$ there is $\pi_1(< a, b >) = a$.
b. $\pi_2 : A \times B \to B$ such that for every $< a, b > \in A \times B$ there is $\pi_2(< a, b >) = b$.

Set of sets F will be called **chain** if for every $A, B \in F$ there is $A \subseteq B$ or $B \subseteq A$. If all the elements in F are functions, F will be called **chain of functions** (*meaning that for every* $f, g \in F$ *there is* $f \subseteq g$ *or* $g \subseteq f$). Very important fact is that union of chain of functions gives a function.

4.6.8 exercise

Given chain of functions F, we'll sign $\tilde{f} := \cup F$. Prove:
a. $\tilde{f}$ is a function, with domain $\cup\{dom(f) : f \in F\}$, and an image $\cup\{im(f) : f \in F\}$.
b. If all the functions in F are injections, then $\tilde{f}$ is also an injection.

4.7 Power Set

Axiom of Power Set: Given any set A, there is a set of all subsets of the set $P(A) : \{B : B \in A\}$.

4.7.1 exercise

Build, in the most shortest way, a set with 128 elements.

4.7.2 exercise

Given any sets A, B. Prove the existence of the set $A \times B$, without using the axiom of Replacement. [*hint: use the set* $P(P(A \cup B))$.]

4.8 Infinity: the Infinite Set and Choice

For every set a, we'll define a's **following** such that: $S(a) = a \cup \{a\}$.

Axiom of Infinity: Exists a set A, such that $\emptyset \in A$ and for every $a \in A$ there is $S(a) \in A$.

This axiom allows us to create an infinite set. At this moment the term "infinite" has not been defined yet, therefore we shall wait till we have all the necessary definitions, and then build the set.
Life without choice can be very boring. When given a collection of a nonempty sets, the next axiom allows to "choose" one element from each set. If the collection is final, we can choose "manually" one element from each set. But the

collection can be too big ("infinite"), so in this case a function that will do the job instead of us is needed.

Axiom of Choice: Given any set S of nonempty sets, there is a function f with $dom(f) = S$, such that for every $A \in S$, $f(A) \in A$.

Function f that satisfies this axiom is called the **function of choice** upon S. When the given collection of nonempty sets S is final, we can prove the existence of the function of choice upon S without the Axiom of Choice.

4.8.1 exercise

Given $A = \{x, y, z\}$ set of nonempty sets. Prove that function of choice upon A exists, without using the axiom of choice.

4.8.2 exercise

Given foreign sets A, B, which elements are nonempty groups. Given f function of choice upon A, and given g function of choice upon B. We'll sign $h := f \cup g$. Prove:
a. h is a function with the domain $A \cup B$.
b. h is a function of choice upon $A \cup B$.
c. Suppose that A, B are not foreign. What must be the condition upon f and g, so that $h = f \cup g$ will be the function of choice upon $A \cup B$?

4.8.3 exercise

Given set F, whose all elements are nonempty and foreign to each other sets. Prove that exists set s, which contains exactly one element from every $X \in F$, and only those elements. [*hint: given f function of choice upon F. Let's look $im(f)$.*]

Set of ordered pairs $R \subseteq A \times A$ will be called **relation upon** A. If $< a, b > /inR$ we'll sign this as aRb, otherwise we'll write $a \not R$. We will also sign $dom(R) = \{a \in A : \exists b \in A \ (aRb)\}$, and $im(R) = \{b \in A : \exists a \in A \ (aRb)\}$.

4.8.4 exercise

Given relation R upon set A. $B = dom(R)$. Prove that exists function f with domain B, such that for every $a \in B$, there is $aRf(a)$. [*hint: use the function of choice upon the collection of all sets from the form $\{b : aRb\}$.*] $\ldots$ and here it ends.

5 Orders

5.1 Definitions

relation R on A is a subset $R \subseteq A \times A$

reflexive $\forall a \in A(aRa)$

anti reflexive $\forall a \in A \neg(aRa)$

symmetric $\forall a, b \in A(aRb \longrightarrow bRa)$

transitive $\forall a, b, c \in A(aRb \wedge bRc \longrightarrow aRc)$

partial/sharp order(P.O) anti-reflexive and transitive.

trichotomy $\forall a, b \in A(aRb \vee bRa \vee a = b)$

linear/hall order partial order and trichotomy.

intuition: we will think of the relation aRb as "b is greater than a". signed: $a <_R b$.

- a is **minimal** in A if $\exists b \in A$ such that $b <_R a$.
- a is **first** (or: smallest) in A if $\forall b \in A$ $a <_R b$ or a=b.
- if R is linear relation on A than $a \in A$ is minimal $\Leftrightarrow$ a is first.

5.2 Partial orders isomorphism

let $< A, R >, < B, S >$ be partial orders. we will say that $< A, R > \cong < B, S >$ if there is a function $f : A \to B$ such that f is 1-1 and on (mapping) such that $\forall a_1, a_2 \in A(a_1 <_R a_2 \to f(a_1) <_S f(a_2))$. we will say that f is **maintaining order**.

intuition: this is weighing relation. it's not really a relation because the elements are cosets , but relation is defined on sets.

- $< A, R >$ is **well order(W.O)** if $< A, R >$ is linear order and $\forall B \subseteq A$, $B \neq \emptyset$ $\exists b \in B$ such that b is first in B.
- example: N - the naturals numbers.
- if $< A, R >$ is P.O, and $a \in A$ than we define $A^a = \{x \in A \mid x <_R a\}$ as **head** of A in a.

Theorem 1 let $< A, R >$ be well order. than:

1. $\forall x, y \in A$ such that $x \leq_R y$ exists $(A^y)^x = A^x$
2. $\forall x \in A, \neg(A^x \cong A)$

Proof

1. $(A^y)^x = \{z \in A^y \mid z <_R x\} = \{z \in A \mid z <_R y \bigwedge z <_R x\} =$ $[x \leq_R y$ therefore $] = \{z \in A \mid z <_R x\} = A^x$
2. let us assume that $A^x \cong A$. let $f : A \to A^x$ be an isomorphism. pay attention that $f(x) <_R x$ because $f(x) \in A^x$. let us look at the set $D = \{a \in A \mid f(a) <_R a\}$ D is not empty because $x \in D$. let a_0 be the first element in D. therefore $f(a_0) <_R a_0$ and $\forall b <_R a_0, b \leq_R f(b)$ especially for $b = f(a_0)$, exists $f(a_0) \leq_R f(f(a_0))$. but $f(a_0) <_R a_0$ and f maintaining order therefore $f(f(a_0)) <_R f(a_0)$. contradiction.

Theorem 2 if $< A, R >, < B, S >$ are isomorphic well orders, than the isomorphism is singular/unique.

Proof Let us assume that $f : A \to B, g : A \to B$ are not the same isomorphism. Therefore $\exists x \in A, f(x) \neq g(x)$. Therefore the set $D = \{x \in A \mid f(x) \neq g(x)\}$ is not empty. let x_0 be the first element in D, i.e $\forall x <_R x_0, f(x) = g(x)$. WLG $f(x_0) <_S g(x_0)$, let us look at $b = g^{-1}(f(x_0))$,notice that $b \neq x_0$ otherwise $g(b) = g(x_0)$, but $g(b) = f(x_0)$ therefore $g(x_0) = f(x_0)$ contradiction to the fact that $x_0 \in D$. notice also that $b >_R x_0$ otherwise (if $b <_R x_0$) $f(b) = g(b) = f(x_0)$ contradiction to 1-1 value of *f*. we just proved that $b >_R x_0$. now, $f(x_0) = g(b) <_S g(x_0)$ but $x_0 <_R b \Rightarrow g$ does not maintaining order. contradiction.

Theorem 3 Let $< A, R >, < B, S >$ be isomorphic W.O, let $f : A \to B$ be the isomorphism. then $\forall a \in A, A^a \cong B^{f(a)}$.

Proof The isomorphism is $f_{|A^a}$.it maintains order and 1-1 value because f is. It's <u>on</u>:let $x \in B^{f(a)}$ then $x <_S f(a)$. f maintains order therefore $f^{-1}(x) <_R a \Rightarrow f^{-1}(x) \in A^a_\square$

5.3 The theorem of good orders isomorphism

Let $< A, R >, < B, S >$ be W.O, then exactly one of the following holds:

1. $< A, R > \cong < B, S >$
2. $\exists a \in A, < A^a, R > \cong < B, S >$
3. $\exists b \in B, < A, R > \cong < B^b, S >$

Proof

singularity 1&2 or 1&3 are impossible. otherwise there exist a W.O that isomorphic to it's on head,and we proved in theorem 1.2 that it is impossible. 2&3 : let $f : B \to A^a$ be an isomorphism. according to 3 $\exists b \in B, A \cong B^b.b \in B \Rightarrow f(b) \in A^a$ therefore $B^b \cong {A^a}^{f(b)} \cong A^{f(b)}$ (because $f(b) <_R a$). now, as result of this $A \cong B^b \cong A^{f(b)}$ but according to theorem 1.2 this is contradiction because $f(b) \in (A^a \subseteq)A$. therefore no two can exist together.

existence we will define $f = \{(x, y) \mid A^x \cong B^y\}$ such that $x \in A, y \in B$.

Lemma 1 f is function.

Proof let us assume f is not a function, i.e $\exists y_1, y_2 \in B$, $y_1 \neq y_2$ and $x \in A$ such that $B^{y_1} \cong A^x \cong B^{y_2}$. WLG $y_2 <_R y_1$ therefore B^{y_2} is head of B^{y_1}.contradiction.

Lemma 2 f is 1-1 value.

Proof like the proof of lemma 1 ,only switching roles. due to symmetric reasons this is correct.

Lemma 3 f maintains order.

Proof let $x_1 <_R x_2 \in dom(f)$, therefore $\exists y_1, y_2 \in B$ such that $A^{x_1} \cong B^{y_1}, A^{x_2} \cong B^{y_2}$. let $g : A^{x_2} \to B^{y_2}$ be the isomorphism. $x_1 <_R x_2 \Rightarrow x_1 \in dom(g) \Rightarrow \exists b = g(x_1)$. therefore $B^{y_1} \cong A^{x_1} \cong B^b$ if $y_1 <_S b$ then B^{y_1} is head of B^b. contradiction. similarly if $b <_S y_1$, therefore $b = y_1$. $b \in B^{y_2}$ therefore $y_1 = b <_S y_2$ $\square$

so far we proved that $dom(f) \cong Im(f)$.

Lemma 4 if $x \in dom(f)$ then $A^x \subseteq dom(f)$. due to symmetric reasons if $y \in Im(f)$ then $B^y \subseteq Im(f)$.

Proof immediate from theorem 3: if $x \in dom(f)$ then $\exists y \in B$ such that $A^x \cong^g B^y$. let $a \in A^x \Rightarrow a <_R x$, then $A^a = A^{x^a} \cong B^{y^{g(a)}} = B^{g(a)}$, therefore $(a, g(a)) \in f$, i.e $a \in dom(f)_\square$.

1. *dom(f)=A*
   (a) $Im(f) = B \Rightarrow A \cong B_{\square(1)}$
   (b) $Im(f) \neq B$: let b be the first element in $B \setminus Im(f) (\neq \emptyset)$. $\forall b_1 \geq_S b, \quad \neg(b_1 \in Im(f))$ otherwise, if $b_1 \in Im(f)$ then according to lemma 4 $B^{b_1} \subseteq Im(f) \Rightarrow b \in Im(f)$ contradiction. b is first in $B \setminus Im(f)$ therefore $\forall b_1 <_S b, b_1 \in Im(f)$. with the above we showed that $Im(f) = B^b$, therefore $A \cong B^b._{\square(2)}$
2. $dom(f) \neq A$ therefore $\exists a \in A$ such that $dom(f) = A^a$. let us assume that $B^b = Im(f)$ for some $b \in B \Rightarrow A^a \cong B^b \Rightarrow (a, b) \in f \Rightarrow a \in dom(f)$ contradiction. therefore $A^a \cong B._{\square(3)}$