## 9. 12.01.06

**Definition 9.1.** We say that a topological space $\langle X, O\rangle$ satisfies $S_1(\mathcal{A},\mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$, there are $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

**Definition 9.2.** We say that a topological space $\langle X, O\rangle$ satisfies $S_{fin}(\mathcal{A},\mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$, there are $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathcal{B}$.

**Definition 9.3.** We say that a topological space $\langle X, O\rangle$ satisfies $U_{fin}(\mathcal{A},\mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$ such that $\mathcal{U}_n$ does not contain a finite cover for all $n \in \mathbb{N}$, there are $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

We will only be interested in $\mathcal{A},\mathcal{B}$ with $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(O)$ and $\bigcup U = X$ for all $U \in \mathcal{A}\cup\mathcal{B}$.

**Observation 9.4.** *Suppose $\langle X, O\rangle$ is a topological space, and $\mathcal{O}$ denotes the family of all open covers of $X$.*

*Then $X \models S_{fin}(\mathcal{A},\mathcal{O})$ iff $X \models U_{fin}(\mathcal{A},\mathcal{O})$.*

*Proof.* Same proof as in Observation 5.15. □

**Observation 9.5** (monotonicity)**.** *If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then $\pi(\mathcal{A}_2,\mathcal{B}_1) \Rightarrow \pi(\mathcal{A}_1,\mathcal{B}_1)$ and $\pi(\mathcal{A}_2,\mathcal{B}_1) \Rightarrow \pi(\mathcal{A}_2,\mathcal{B}_2)$, where $\pi \in \{S_1, S_{fin}, U_{fin}\}$.*

**Lemma 9.6.** *Suppose $\langle X, O\rangle$ is a Lindelöf topological space, $\mathcal{B} \subseteq \mathcal{P}(O)$, and let $\Gamma := \Gamma_X$ denote the family of all $\gamma$-covers of $X$.*[29]

*Then $X \models U_{fin}(\Gamma,\mathcal{B})$ iff for all $\mathcal{A}$, a family of open covers of $X$, $X \models U_{fin}(\mathcal{A},\mathcal{B})$.*

*Proof.* We would like to prove:

$$\forall \mathcal{A}. X \models U_{fin}(\mathcal{A},\mathcal{B}) \Rightarrow X \models U_{fin}(\Gamma,\mathcal{B}) \Rightarrow X \models U_{fin}(\mathcal{O},\mathcal{B}) \Rightarrow \forall \mathcal{A}. X \models U_{fin}(\mathcal{A},\mathcal{B}).$$

But the only non-trivial implication is $X \models U_{fin}(\Gamma,\mathcal{B}) \Rightarrow X \models U_{fin}(\mathcal{O},\mathcal{B})$.

Assume $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$ are given and no $\mathcal{U}_n$ contains a finite cover. Fix $n \in \mathbb{N}$. By Lindelöfness, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$. Let $\mathcal{V}_n := \{V_n^k \mid k \in \mathbb{N}\}$ where $V_n^k := \bigcup_{m\leq k} U_n^m$ for all $k \in \mathbb{N}$. Since $\mathcal{U}_n$ contains no finite cover, we know that $\mathcal{V}_n \in \Gamma$.

By $X \models U_{fin}(\Gamma,\mathcal{B})$, there exists $f : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ such that if we let $\mathcal{F}_n := \{V_n^k \mid k \in f(n)\}$ for all $n \in \mathbb{N}$, then $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

Define $g : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ by letting $g(n) := \{m \in \mathbb{N} \mid \exists k \in f(n). m \leq k\}$ for all $n \in \mathbb{N}$. It is evident that $\bigcup \mathcal{G}_n = \bigcup \mathcal{F}_n$ whenever $n \in \mathbb{N}$ and $\mathcal{G}_n := \{U_n^k \mid k \in g(n)\} \in [\mathcal{U}_n]^{<\omega}$. □

**Corollary 9.7.** *Suppose $\langle X, O\rangle$ is a Lindelöf topological space. Let $\Gamma := \Gamma_X$.*

*Then $X \models S_{fin}(\mathcal{O},\mathcal{O})$ iff $X \models U_{fin}(\mathcal{O},\mathcal{O})$ iff $X \models U_{fin}(\Gamma,\mathcal{O})$*

[29] Recall Definition 5.8.

**Corollary 9.8.** *Suppose $\langle X, O\rangle$ is a Lindelöf topological space. Let $\Gamma := \Gamma_X$.*
*Then $X \models U_{fin}(\mathcal{O}, \Gamma)$ iff $X \models U_{fin}(\Gamma, \Gamma)$.*

**Proposition 9.9.** *$S_1(\mathcal{O}, \Gamma)$ is trivial*

*Proof.* Because it implies $S_{fin}(\mathcal{O}, \Gamma)$. Now recall Observation 5.14. □

The same trick of the proof of Theorem 9.6 will prove that $S_1(\Gamma, \Gamma)$ implies $U_{fin}(\mathcal{O}, \Gamma)$ and that $S_1(\Gamma, \mathcal{O})$ implies $S_{fin}(\mathcal{O}, \mathcal{O})$, thus we obtain the following diagram of implications:

$$\begin{array}{ccc}
U_{fin}(\mathcal{O},\Gamma) & \longrightarrow & S_{fin}(\mathcal{O},\mathcal{O}) \\
\uparrow & & \uparrow \\
S_1(\Gamma,\Gamma) & \longrightarrow & S_1(\Gamma,\mathcal{O}) \\
 & & \uparrow \\
 & & S_1(\mathcal{O},\mathcal{O})
\end{array}$$

**Theorem 9.10** (Scheepers-Just-Miller-Szeptycki)**.** *$S_{fin}(\Gamma, \Gamma) = S_1(\Gamma, \Gamma)$.*

*Proof.* Suppose $\langle X, O\rangle$ is a topological space, $\Gamma := \Gamma_X$, and $X \models S_{fin}(\Gamma, \Gamma)$.

Assume $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \Gamma$. By Observation 5.9, if we pick $\langle U_n \in \mathcal{F}_n \mid n \in \mathbb{N}\rangle$, then also $\{U_n \mid n \in \mathbb{N}\} \in \Gamma$ and we are almost done.

In order to be done, we need to somehow ensure that we indeed selected an element $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$, but this wouldn't happen in the above approach if there exists empty $\mathcal{F}_n$'s. To complete the proof, we need the following. □

We now generalize the idea of Observation 7.13.

**Lemma 9.11.** *Suppose $\langle X, O\rangle$ is a topological space, then $X \models S_1(\Gamma, \Gamma)$ iff for all $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{\leq 1} \mid n \in \mathbb{N}\rangle$ such that $\{U \mid \exists n \in \mathbb{N}(U \in \mathcal{F}_n)\} \in \Gamma$.*

*Proof.* Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n := \{U_1^k \cap ... \cap U_n^k \mid k \in \mathbb{N}\}$. Clearly, $\langle \mathcal{V}_n \mid n \in \mathbb{N}\rangle$ is a sequence of $\gamma$-covers, so by the hypothesis we find $\mathcal{F}_n \in [\mathcal{V}_n]^{\leq 1}$ for each $n \in \mathbb{N}$.

Let $f : \mathbb{N} \to \mathbb{N} \cup \{\star\}$ be the function such that for all $n \in \mathbb{N}$, $f(n) = \{\star\}$ if $\mathcal{F}_n = \emptyset$, and $\mathcal{F}_n = \{U_1^{f(n)} \cap ... \cap U_n^{f(n)}\}$, otherwise. Since $\{U \mid \exists n \in \mathbb{N}(U \in \mathcal{F}_n)\} \in \Gamma$, $\mathrm{Im}(f)$ is infinite, and the function $\tilde{f} : \mathbb{N} \to \mathbb{N}$ is well-defined:

$$f(n) := \{f(m) \mid m = \min\{k \geq n \mid f(k) \neq \star\}.$$

For $n \in \mathbb{N}$, put $U_n := U_n^{\tilde{f}(n)}$. It is now obvious that $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ is a witness to $S_1(\Gamma, \Gamma)$.[30] □

**Observation 9.12.** *Assume $\langle X, O\rangle$ is a topological space, and $\langle U_n \mid n \in \mathbb{N}\rangle$ is a **sequence** of open sets such that $\{n \in \mathbb{N} \mid x \notin U_n\}$ is finite for all $x \in X$.*

*If $X \neq U_n$ for all $n \in N$, then $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\}$ is an infinite **set**, and in particular $\mathcal{U} \in \Gamma$.*

*Proof.* Suppose not, then by a trivial pigeonhole argument, there exists some $m \in \mathbb{N}$ and infinite $I \subseteq \mathbb{N}$ such that $U_n = U_m$ for all $n \in I$. Since $U_n \neq X$, we may pick $x \in X \setminus U_m$ and conclude that $I \subseteq \{n \in \mathbb{N} \mid x \notin U_n\}$, yielding a contradiction to the finiteness hypothesis. □

[30] More accurately, it is a witness to an instance of $S_1(\Gamma, \Gamma)$, because the family $\langle U_n \mid n \in \mathbb{N}\rangle$ were already given.