8. 05.01.06

Observation 8.1 (ZFC+BC). *If $\langle X, d\rangle$ is metric, and $X \models S_1(\mathcal{O},\mathcal{O})$, then $|X| \leq \aleph_0$.*

Proof. By Observation 7.14, every $S_1(\mathcal{O},\mathcal{O})$ metric space is strongly null. Thus, if Borel's conjecture 7.5 holds, then X must be countable. □

If one omits the requirement of metricity, we get the following.

Theorem 8.2 (ZFC). *There exists an unctounable non-metrizable space that satisfies $S_1(\mathcal{O},\mathcal{O})$.*

Proof. Consider $X := \omega_1 + 1$. We equip X with the *interval topology*. Let $\langle X, O\rangle$ be the topological space determined by the base:

$$B := \{\alpha^{\uparrow}, \alpha^{\downarrow}, (\beta,\alpha) \mid \beta < \alpha < \omega_1\},$$

where $\alpha^{\uparrow} := \{\gamma \in X \mid \gamma > \alpha\}$, $\alpha^{\downarrow} := \{\gamma \in X \mid \gamma < \alpha\}$, $(\beta,\alpha) := \beta^{\uparrow} \cap \alpha^{\downarrow}$. We now show that X is concentrated on the singleton $\{\omega_1\}$, concluding that $X \models S_1(\mathcal{O},\mathcal{O})$. Indeed, if U is an open set containing ω_1, then $U \supseteq \alpha^{\uparrow}$ for some $\alpha < \omega_1$. For such α, we get that $(X \setminus U) \subseteq \alpha + 1$, and in particular, $(X \setminus U)$ is countable. □

We now work towards giving a direct proof to Corollary 7.33.

Lemma 8.3 (Embedding). *Suppose there is a dominating/unbouned/strongly-unbounded family of cardinality κ, and $A \subseteq \{0,1\}^{\omega}$ is a set of cardinality $\leq \kappa$.*

Then, there there exists a set $B \in [\omega^{\omega}]^{\kappa}$ and a continuous function $\phi : \omega^{\omega} \to \omega^{\omega}$ such that B is dominating/unbouned/strongly-unbounded (respectively), and $\phi[B] = A$.

Proof. Assume $D = \{f_\alpha \mid \alpha < \kappa\} \in [\omega^{\omega}]^{\kappa}$ is unbounded (or dominating, or strongly-unbounded). Let $\{g_\alpha \mid \alpha < \kappa\}$ enumerate A. Put $B := \{h_\alpha \mid \alpha < \kappa\}$, where:

$$h_\alpha(n) := 2f_\alpha(n) + g_\alpha(n) \qquad (\alpha < \kappa, n < \omega)$$

B is evidently unbounded (or dominating, or strongly-unbounded). Finally, define a continuous function $\phi : \omega^{\omega} \to \omega^{\omega}$ by letting for all $f \in \omega^{\omega}$ and $n < \omega$: $\phi(f)(n) = f(n) \mod 2$. □

Lemma 8.4 (Interleaving). *Suppose there is an unbouned/strongly-unbounded family of cardinality κ, and $A \subseteq \omega^{\omega}$ is a set of cardinality $\leq \kappa$.*

Then, there there exists a set $B \in [\omega^{\omega}]^{\kappa}$ and a continuous surjection $\phi : \omega^{\omega} \to \omega^{\omega}$ such that B is unbouned/strongly-unbounded (respectively), and $\phi[B] = A$.

Proof. Assume $D = \{f_\alpha \mid \alpha < \kappa\} \in [\omega^{\omega}]^{\kappa}$ is unbounded (or strongly-unbounded). Let $\{g_\alpha \mid \alpha < \kappa\}$ enumerate A. Put $B := \{h_\alpha \mid \alpha < \kappa\}$, where:

$$h_\alpha(n) = \begin{cases} f_\alpha(k) & \exists k < \omega(n = 2k) \\ g_\alpha(k) & \exists k < \omega(n = 2k+1) \end{cases}$$

B is evidently unbounded (or strongly-unbounded). Finally, define $\phi : \omega^\omega \to \omega^\omega$ in the obvious way. □

Definition 8.5. Assume κ is a cardinal, and $\mathcal{I}$ is an ideal over some set X.

We say that $\mathcal{I}$ has the *κ-flexability* property iff $\mathcal{I}$ is non-trivial, and whenever $Y \subseteq X$ is κ-concentrated on some $A \in \mathcal{I}$, then $Y \in \mathcal{I}$.

Observation 8.6. *Suppose $\mathcal{I}$ is an ideal over some set X that has the κ-flexability property, then $\mathsf{non}(\mathcal{I}) \geq \kappa$.*

Proof. Fix $A \in [X]^{<\kappa}$. Pick $a \in A$. Since $\mathcal{I}$ is non-trivial, $\{a\} \in \mathcal{I}$. It is now obvious that A is κ-concentrated at $\{a\} \in \mathcal{I}$. □

Observation 8.7. *$\mathcal{N}$ has the $\mathsf{non}(\mathcal{N})$-flexability property.*

$\mathcal{SN}$ has the $\mathsf{non}(\mathcal{SN})$-flexability property.

Proof. Assume Y, A are subsets of $\mathbb{R}$, where $A \in \mathcal{I}$ and Y is $\mathsf{non}(\mathcal{N})$-concentrated at A.

Fix $\varepsilon > 0$. Since $A \in \mathcal{I}$, we may find a family of open sets $\{U_n \mid n \in \mathbb{N}\}$ with $\sum_{n\in\mathbb{N}} \mathrm{Diam}(U_n) < \frac{\varepsilon}{2}$, and $A \subseteq U := \bigcup_{n\in\mathbb{N}} U_n$

Since U is open containing A, $|Y \setminus U| < \mathsf{non}(\mathcal{N})$. In particular, $(Y \setminus U) \in \mathcal{N}$ and we may find a family of open sets $\{V_n \mid n \in \mathbb{N}\}$ such that $(Y \setminus U) \subseteq \bigcup_{n\in\mathbb{N}} V_n$ and $\sum_{n\in\mathbb{N}} \mathrm{Diam}(V_n) < \frac{\varepsilon}{2}$.

The proof for the case of $\mathcal{SN}$ is essentially the same. □

Theorem 8.8. *Assume $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ is a non-trivial, σ-additive, proper ideal.*

Then for any ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ and a cardinal $\kappa \geq \mathsf{non}(\mathcal{J})$ such that:

- *$\mathcal{I}$ has the κ-flexability property;*
- *There exists a strongly-unbounded family of size κ.*

there exists $X \in \mathcal{I}$, and a continuous function $f : X \to \mathbb{R}$ such that $f[X] \notin \mathcal{J}$.

Proof. Pick $A \in [\mathbb{R}]^{\mathsf{non}(\mathcal{J})}$, with $A \notin \mathcal{J}$. If $\{A \cap [z, z+1] \mid z \in \mathbb{Z}\} \subseteq \mathcal{J}$, then by the σ-additivity of $\mathcal{J}$, $A \in \mathcal{J}$. It follows that there exists $z \in \mathbb{Z}$, such that $[z, z+1] \cap A \notin \mathcal{J}$.

For notational simplicity, we assume $A \subseteq [0,1]$. $\mathcal{J}$ is σ-additive and non-trivial, thus $\mathbb{Q} \in \mathcal{J}$, hence, we may also assume that $A \cap \mathbb{Q} = \emptyset$.

Altogether, we assume $A \subseteq ([0,1] \setminus \mathbb{Q})$, $|A| = \mathsf{non}(\mathcal{J})$, and $A \notin \mathcal{J}$.

Let $\psi : [0,1] \setminus \mathbb{Q} \to \omega^\omega$ be an homeomorphism. Put $A' := \psi[A]$. By the interleaving lemma 8.4, there exists a strongly-unbounded $B \in [\omega^\omega]^\kappa$, and a continuous function $\phi : \omega^\omega \to \omega^\omega$ such that $\phi[B] = A'$. Let $X := \psi^{-1}[B]$ and $f := (\psi^{-1} \circ \phi \circ \psi) \restriction X$.

Notice that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ is a composition of continuous functions, and:

$$f[X] = \psi^{-1}[\phi[\psi[X]] = \psi^{-1}[\phi[B]] = \psi^{-1}[A'] = A \notin \mathcal{J}.$$

We are left with showing that $X \in \mathcal{I}$. Since $\mathcal{I}$ satisfies the κ-flexability property, it suffices to show that X is κ-concentrated at some set from $\mathcal{I}$. By Observation 8.6 and the hypothesis, $\mathsf{non}(\mathcal{I}) \geq \kappa \geq \mathsf{non}(\mathcal{J}) \geq \mathsf{add}(\mathcal{J}) \geq \aleph_1$, thus $\mathbb{Q} \in \mathcal{I}$. Finally, notice that if U is an open set containing $\mathbb{Q}$, then $\psi[[0,1] \setminus U]$ is compact, thus $\leq^*$-bounded, thus $\psi[X \setminus U]$ is a $\leq^*$-bounded subset of the strongly-unbounded set B, and hence, $|X \setminus U| = |\psi[X \setminus U]| < |B| = \kappa$. □

Thus, for instance, if CH holds, we may find a strongly-null subset of $\mathbb{R}$ with a continuous image which is not null. We may also find a strongly-null subset of $\mathbb{R}$ with a continuous image which is not meager. In particular, this set must be uncountable, thus we had obtained an alternative proof to the fact that $\mathsf{CH} \Longrightarrow \neg\mathsf{BC}$.

Proposition 8.9 (CH)**.** *Assume that $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is an ideal that has the $\aleph_1$-flexability property, then for any $Y \subseteq \omega^\omega$, there exists $X \in \mathcal{I}$ and a continuous $f : X \to \omega^\omega$ such that $f[X] = Y$.*

Proof. Fix $Y \subseteq \mathbb{R}$. If Y is countable, this is easy (recall Observation 8.6).

Assume that Y is uncountable. By CH, we may fix a $\mathfrak{b}$-scale $\{f_y \in \omega^\omega \mid y \in Y\}$. Now, by applying the interleaving lemma 8.4, we obtain a set $B \subseteq \omega^\omega$ that interleaves ω^ω inside this scale. In greater details, we obtain a strongly-unbounded set B of size $\mathfrak{b}$, and a continuous function $\phi : \omega^\omega \to \omega^\omega$ such that $\phi[B] = Y$. Let $\psi : [0,1] \setminus \mathbb{Q} \to \omega^\omega$ be an homeomorphism.

Put $X := \psi^{-1}[B]$ and $f = (\phi \circ \psi) \restriction X$. Evidently, f is continuous and $f[X] = Y$.

The standard argument shows that X is $\mathfrak{b}$-concentrated at $\mathbb{Q}$. Finally, it follows from the hypothesis that $\mathbb{Q} \in \mathcal{I}$, $\mathfrak{b} = \aleph_1$ and $X \in \mathcal{I}$. □

Corollary 8.10 (CH)**.** *There exists $X \in \mathcal{SN}$, and a continuous function $f : X \to \mathbb{R}$ such that $f[X] \in \mathcal{SN}^*$, i.e., a strongly-null set whose continuous image is of Lebesgue measure 1.*

Proof. Since $(0,1) \setminus \mathbb{Q}$ is of Lebesgue measure 1 and a continuous image of ω^ω. □

It is worth mentioning that one can prove in ZFC that there exists continuous mapping from the cantor set (=a set of measure zero) onto the unit interval (=a set of measure 1).

Question 8.11. Suppose there exists an arbitrary metric space $\langle X, d\rangle$ which is uncountable and strongly-null, must this indicate the violation of Borel's Conjecture 7.5 ?

Question 8.12 (Miller)**.** Suppose there exists a metric space $\langle X, d\rangle$ which is strongly-null and $|X| = \mathfrak{c}$, must this indicate the existence of $Y \in [\mathbb{R}]^{\mathfrak{c}}$ which is SMZ ?

The second question is unsolved. We shall now work towards introducing a positive answer to the first question. The key to the solution of this question is Carlson's lemma. 8.21 which is deeply inspired by Urysohn's Theorem 8.20.

Definition 8.13. A topological space $\langle X, O\rangle$ is T_1 iff $\{x\}$ is a closed subset for all $x \in X$.

Definition 8.14. A T_1 topological space X is *regular* iff whenever A is closed subset of X and $x \notin A$, then there are disjoint open sets U, V with $x \in U$ and $A \subseteq V$.

A T_1 topological space X is *normal* iff whenever A, B are disjoint closed sets in X, then there are disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$.

Notice that a metric space is normal and regular. Actually, we had already took advantage of this property in the proof of Theorem 3.16. Also notice that a normal space is regular, since in a T_1 space points are closed sets.

Observation 8.15. *Suppose $\langle X, O\rangle$ is a topological space such that for any two closed subsets A, B, there exists a continuous function $f : X \to [0, 1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$, then X is normal.*

Proof. Fix closed subsets A, B, and let f be like in the hypothesis. Then $f^{-1}[0, 0.5)$ and $f^{-1}(0.5, 1]$ are mutually disjoint open sets, containing A and B respectively. $\square$

Urysohn, in his celebrated lemma, was able to prove the converse:

Lemma 8.16 (Urysohn)**.** *Let X be a normal topological space, and $A, B \subset X$ are disjoint and closed. Then there exist a continuous function $f : X \to [0, 1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.*

Proof. Fix an enumeration $\mathbb{Q} \cap [0, 1] = \{r_n \mid n \in \mathbb{N}\rangle$ with $r_1 = 1$ and $r_2 = 0$. We will construct a family of open sets $\langle V_r | r \in \mathbb{Q} \cap [0, 1]\rangle$ by induction on $n \in \mathbb{N}$. The family will satisfy:

$$r < r' \Longrightarrow \overline{V_r} \subset V_{r'} \qquad (r, r' \in \mathbb{Q} \cap [0, 1])$$

Inductinon base $n \in \{1, 2\}$: Put $V_1 = V_{r_1} := B^c$. Since X is normal, the separation $A \subseteq U \subseteq \overline{U} \subset B^c$, where U is open, is possible. Pick such U and let $V_0 = V_{r_2} := U$.

Inductive hypothesis: Assume we had already defined $V_{r_1}, V_{r_2}, ..., V_{r_n}$.

Induction step $n + 1$: Find $m, l \in \mathbb{N}$ such that $r_m := \max\{r_i \mid i \le n, r_i < r_n\}$ and $r_l := \min\{r_i \mid i \le n, r_i > r_n\}$ (”closest” rationals to r_{n+1} so far). By the normality of X, an open set U exists such that $V_{r_m} \subseteq U \subset \overline{U} \subseteq V_{r_l}$. Define $V_{r_{n+1}} := U$. End of the construction.

We now define a function $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} \inf\{r | x \in V_r\} & \text{if } x \in V_1 \\ 1 & \text{if } x \in B \end{cases}$$

In order to prove that f is continuous, it is suffice to show that $f^{-1}[0,a)$ and $f^{-1}(b,1]$ are open subsets of X for any $a,b\in\mathbb{R}$. Indeed:

$$f^{-1}[0,a)=\{x|f(x)<a\}=\{x|\exists r\in\mathbb{Q},r<a,x\in V_r\}=\bigcup_{\substack{0\leq r<a\\ r\in\mathbb{Q}}}V_r.$$

This is a union of open sets, thus open.

$$f^{-1}(b,1]=\{x|f(x)>b\}=\{x|f(x)\leq b\}^c=\{x|\forall r'>b,x\in V_{r'}\}^c=\{x|\exists r'>b,x\notin V_{r'}\}=$$
$$\{x|\exists r,r',r'>r>b,x\notin\overline{V_r}\subseteq V_{r'}\}=\bigcup_{b<r\leq 1}\overline{V_r}^c.$$

Again, this is a union of open sets, hence open. □

In order to prove our next theorem we will have to introduce the Hilbert space ℓ_2.

Definition 8.17. A natural extension of finite dimensional euclidian spaces is

$$\ell_2:=\Big\{(x_1,x_2,...)\mid x_i\in\mathbb{R},\sum_{n\in\mathbb{N}}x_n^2<\infty\Big\}.$$

For any two elements $x,y\in\ell_2$, the inner product is defined by $\langle x,y\rangle:=\sum_{n\in\mathbb{N}}x_ny_n$. It is well known that any inner product space is a normed space by defining

$$||x-y||^2:=\langle x-y,x-y\rangle \qquad (x,y\in\ell_2)$$

Notice that ℓ_2 is separable. A countable dense set is $\big\{(x_1,...,x_n,0,0,..)\in\ell_2\mid n\in\mathbb{N},x_i\in\mathbb{Q}\big\}$.

Theorem 8.18 (Urysohn)**.** *A second countable normal space is metrizable.*[26]

Proof. Let X be a second countable normal space, and assume $\mathcal{B}=\{B_j\mid j\in\mathbb{N}\}$ is a countable base for the topology on X. Put $\mathcal{I}:=\big\{(j,i)\in\mathbb{N}\times\mathbb{N}\mid\overline{B_j}\subseteq B_i\big\}$.

For each $(j,i)\in\mathcal{I}$, by applying Urysohn's lemma 8.16, we may pick a continuous function $f_{j,i}:X\to[0,1]$ such that $f_{j,i}[B_i^c]=\{1\}$ and $f_{j,i}[\overline{B_j}]=\{0\}$. Let us enumerate these functions $\langle f_{j,i}\mid(j,i)\in\mathcal{I}\rangle=\langle g_n\mid n\in\mathbb{N}\rangle$ and define a function $G:X\to\ell_2$ by letting for each $x\in\ell_2$:

$$G(x):=\Big(g_1(x),\frac{g_2(x)}{2},...,\frac{g_n(x)}{n},...\Big).$$

Showing that G is a homeomorphism on $G[X]\subseteq\ell_2$ will do, since a subspace of a metrizable space is metrizable.

G is an injection: Fix $x\neq y$ in X. It suffices to find $(j,i)\in\mathcal{I}$ such that $f_{j,i}(x)\neq f_{j,i}(y)$. X is T_1, thus a base set $B_i\in\mathcal{B}$ exists, such that $x\in B_i$ and $y\notin B_i$. Now, since X is normal, a base set $B_j\in\mathcal{B}$ exists, such that $x\in B_j\subseteq\overline{B_j}\subseteq B_i$, hence, $f_{j,i}(x)=1$ and $f_{j,i}(y)=0$.

[26]Recall that a *second countable* topological space is a space with a countable base to its' topology.

G is continuous: Let $x \in X$ and $\varepsilon > 0$. Let N be large enough so that $\sum_{n>N} \frac{1}{n^2} < \varepsilon^2$. The functions $g_1, ..., g_N$ are continuous, therefore there are open sets $U_1, ..., U_N$, containing x, such that $\frac{1}{n^2}|g_n(x) - g_n(x_n)|^2 < \frac{\varepsilon^2}{N}$ whenever $1 \leq n \leq N$ and $x_n \in U_n$. Finally, for every $u \in U := \bigcap_{1 \leq n \leq N} U_n$, we have:

$$||G(x) - G(u)||^2 = \sum_{n \in \mathbb{N}} \frac{|g_n(x) - g_n(u)|}{n^2} < 2\varepsilon^2.$$

We get that for every $x \in X$ there existt an open set $x \in U$ such that $G(U) \subseteq \mathsf{B}_{\sqrt{2}\varepsilon}(G(x))$, that is G is continuous.

G is open: Let U be an open subset of X and pick $x \in U$. Since X is regular there are $B_i, B_j \in \mathcal{B}$ such that $x \in B_j \subseteq \overline{B_j} \subseteq B_i$.

Now, $g_n = f_{j,i}$ satisfies $g_n(x) = 0$ and $g_n(U^c) = 1$, therefore, for $y \in U^c$

$$||G(x) - G(y)|| \geq \frac{1}{n^2}|g_n(x) - g_n(y)|^2 = \frac{1}{n^2}.$$

We get that if y satisfies $G(y) \in \mathsf{B}_{\frac{1}{2n}}(G(x))$ than $y \notin U^c$, meaning that $y \in U$ and therefore $\mathsf{B}_{\frac{1}{2n}}(G(x)) \subset G(U)$, hence G is open. □

The previous theorem can be strengthened with some more topological arguments.

Lemma 8.19. *A second countable regular space X is normal.*

Proof. Suppose A and B are mutually-disjoint closed subsets of X.

Assume $\mathcal{B} = \langle D_n | n \in \mathbb{N} \rangle$ is a countable base to X. Fix functions $f : A \to \mathbb{N}, g : B \to \mathbb{N}$ such that:

- For all $x \in A$: $x \in D_{f(x)} \subseteq \overline{D_{f(x)}} \subseteq B^c$;
- For all $y \in B$: $y \in D_{g(y)} \subseteq \overline{D_{g(y)}} \subseteq A^c$.

To see such function exists, fix for instance $x \in A$. Since X is regular, a base set $D_n \in \mathcal{B}$ exists such that $x \in D_n \subseteq \overline{D_n} \subseteq B^c$.

Enumerate $\{U_n \mid n \in \mathbb{N}\} = \{D_{f(x)} \mid x \in A\}$ and $\{V_n \mid n \in \mathbb{N}\} = \{D_{g(y)} \mid y \in B\}$. It follows that $A \subseteq \bigcup_{n \in \mathbb{N}} U_n, B \subseteq \bigcup_{n \in \mathbb{N}} V_n$, and $B \cap \overline{U_n} = \emptyset, A \cap \overline{V_n} = \emptyset$ for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, define $U'_n := U_n \setminus \bigcup_{i \leq n} \overline{V_i}$ and $V'_n := V_n \setminus \bigcup_{i \leq n} \overline{U_i}$.

Notice that $U := \bigcup_{n \in \mathbb{N}} U'_n$ is a union of open sets, thus open. Same for $V := \bigcup_{n \in \mathbb{N}} V'_n$.

Also, by the choice of $\{U_n, V_n \mid n \in \mathbb{N}\}$, $A \subseteq U$ and $B \subseteq V$. We are left with showing that $U \cap V = \emptyset$. Assume that there is x with $x \in U \cap V$, that is, there are $i, j \in \mathbb{N}$ with $x \in U'_i \cap V'_j$. Obviously, $i \neq j$. Actually, if $i < j$, then $x \notin V'_j$, and if $i > j$, then $x \notin U'_i$. Altogether, we get that $U \cap V = \emptyset$. □

Corollary 8.20 (Urysohn)**.** *A second countable regular space is metrizable.*

ℓ_2 is a separable metric space. Urysohn's theorem assures us that a second countable regular space is separable and metrizable. On the other hand, any separable metrizable space is second countable[27] and normal (hence regular), thus the equivalence. knowing that, we get that every separable metrizable space is homeomorphic to some subspace of ℓ_2.

Lemma 8.21 (Carlson). *If $\langle X, d\rangle$ is a separable metric space and $|X| < 2^{\aleph_0}$, then there exists an injection $\psi : X \to \mathbb{R}$ such that $|\psi(x) - \psi(y)| \leq d(x, y)$ for all $x, y \in X$.*

Proof. By Lemma 5.3, we may assume that $\mathrm{Im}(d) \subseteq [0, 1]$.[28] Since X is separable, we may pick a dense subset $\{x_n \mid n < \omega\}$. For each $x \in X$, attach an analytic function on the unit ball, $f_x : \{y \in \mathbb{C} \mid |y| < 1\} \to \mathbb{C}$, by letting:

$$f_x(z) := \sum_{n=0}^{\infty} \frac{d(x, x_n)}{n!} z^n.$$

Since $x \mapsto \langle d(x, x_n) \mid n < \omega\rangle$ is one-to-one, and two analytic functions with different Taylor expension are different, we have that $x \mapsto f_x$ is one-to-one.

Lemma 8.22. *If f, g are two analytic functions, then $A_{f,g} := \{z \mid f(z) = g(z)\}$ is countable.*

Proof. Suppose not, then we could find a compact subset $K \subseteq \mathbb{C}$ such that $K \cap A_{f,g}$ is uncountable. In particular, f and g are two analytic functions that share an accumulation point, and we must have conclude that $f = g$. □

Put $A := \bigcup\{A_{f_x,f_y} \mid x, y \in X, x \neq y\}$. $|A| < 2^{\aleph_0}$ since $|X| < 2^{\aleph_0}$, and it follows that we may pick $r \in [0, \ln(e)] \subseteq \mathbb{R}$ such that $r \notin A$. Define $\psi : X \to \mathbb{R}$ by $\psi(x) := f_x(r)$ for all $x \in X$. ψ is an injection. To see that it satisfies the Lipshitz property, notice that for all $x, y \in X$, we have:

$$|\psi(x)-\psi(y)| = |f_x(r)-f_y(r)| = \Big|\sum_{n=0}^{\infty} \frac{d(x, x_n)}{n!} r^n - \sum_{n=0}^{\infty} \frac{d(y, x_n)}{n!} r^n\Big| = \Big|\sum_{n=0}^{\infty} \frac{d(x, x_n) - d(y, x_n)}{n!} r^n\Big|$$

$$\leq \sum_{n=0}^{\infty} \frac{d(x, y)}{n!} r^n = e^r \cdot d(x, y) \leq e^{\ln(e)} \cdot d(x, y) = d(x, y).$$

□

Theorem 8.23 (Carlson). *If there exists an uncountable metric space which is strongly null, then $\neg\mathsf{BC}$.*

[27] Consider all open balls of rational radiuses centered at elements of a countable dense set.

[28] Notice that if $\langle X, d\rangle$ is strongly null, then so is $\langle X, \frac{d}{1+d}\rangle$.

Proof. If $\mathfrak{c} = \aleph_1$, then by corollaries 3.9 and 7.15, $\neg\mathsf{BC}$ and we are done. Assume $\mathfrak{c} > \aleph_1$. Assume that $\langle X, d\rangle$ is an uncountable strongly-null metric space, then for all $Y \in [X]^{\aleph_1}$, $\langle Y, d\rangle$ is a strongly-null metric space of cardinality $< 2^{\aleph_0}$. Had we known that Y is separable, we could use Lemmas 8.21,7.8 to complete the proof. Recalling Lemma 2.6, we are left with proving the following. □

Lemma 8.24. *Assume $\langle X, d\rangle$ is a strongly null metric space, then X is second-countable.*

Proof. By the hypothesis, for all $n \in \mathbb{N}$, we may find $\langle x^n_m \in X \mid m \in \mathbb{N}\rangle$ and $\{\varepsilon^m_n \in (0, \infty) \mid m \in \mathbb{N}\}$ such that $X \subseteq \bigcup_{m\in\mathbb{N}} \mathsf{B}_{\varepsilon^n_m}(x^n_m)$ and $\sum_{m\in\mathbb{N}} \varepsilon^n_m < \frac{1}{n}$. A moment's reflection makes it clear that $\{\mathsf{B}_{\varepsilon^n_m}(x^n_m) \mid n, m \in \mathbb{N}\}$ is a base to X. □

Corollary 8.25. *Suppose $\langle X, d\rangle$ is a metric space and $X \models S_1(\mathcal{O}, \mathcal{O})$, then $w(X) = \aleph_0$.*

Proof. By Observation 7.14 and the preceding lemma. □

Definition 8.26. Suppose $\langle X, O\rangle$ is a topological space, let $o(X) = |O| + \aleph_0$.

Corollary 8.27. *Suppose $\langle X, d\rangle$ is a metric space and $X \models S_1(\mathcal{O}, \mathcal{O})$, then $o(X) \leq w(x)^{\aleph_0}$.*

Proof. By the preceding Lemma, we may pick a base $\mathcal{B}$ of cardinality $\aleph_0$, and then any $U \in O$ is of the form $U = \bigcup \mathcal{U}$ for some $\mathcal{U} \subseteq \mathcal{B}$, i.e., $U = \bigcup \mathcal{U}$ for some $\mathcal{U} \in [\mathcal{B}]^{\leq\aleph_0}$. □

We now work towards proving the same for $S_{fin}(O, O)$.

Lemma 8.28. *Suppose $\langle X, d\rangle$ is a metric space, then any open set U is F_σ.*

Proof. Since U is open $U = \bigcup_{i\in I} \mathsf{B}_{r_i}(x_i)$ (where I is some index set and $\mathsf{B}_{r_i}(x_i)$ is an open ball of radius r_i centered at x_i).

For every $i \in I$ fix some sequence $\langle \varepsilon_{i_k} \mid k \in \mathbb{N}\rangle$ such $\varepsilon_{i_k} \to r_i$. Define $F_k := \bigcup_{i\in I} \overline{\mathsf{B}_{\varepsilon_{i_k}}(x_i)}$.

Evidently $U = \bigcup_{k\in\mathbb{N}} F_k$.

□

Lemma 8.29. *The property $S_{fin}(\mathcal{O}, \mathcal{O})$ is σ-additive.*

Proof. Suppose $\langle X, O\rangle$ is a metric space, and $\langle X_m \subseteq X \mid m < \omega\rangle$ is a family of subspaces, each satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$. We shall show that $\bigcup_{m\in\mathbb{N}} X_m \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Assume $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ is a family of open covers of $\bigcup_{n\in\mathbb{N}} X_n$. Put $\mathbb{N} = \biguplus_{m\in\mathbb{N}} A_m$ where each A_m is infinite. For $m \in \mathbb{N}$, by $X_m \models S_{fin}(\mathcal{O}, \mathcal{O})$, we may find $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in A_m\rangle$ such that $X_m \subseteq \bigcup\bigcup_{n\in A_m} \mathcal{F}_m$. It follows that $\bigcup_{m\in\mathbb{N}} X_m \subseteq \bigcup\bigcup_{m\in\mathbb{N}} \mathcal{F}_m$. □

Corollary 8.30. *$S_{fin}(\mathcal{O}, \mathcal{O})$ is open hereditary to any metric space.*

Proof. By Observation 1.27, $S_{fin}(\mathcal{O}, \mathcal{O})$ is closed hereditary. Now apply Lemmas 8.28,8.29.
□

Corollary 8.31. *Suppose* $\langle X, d\rangle$ *is a metric space and* $X \models S_{fin}(\mathcal{O}, \mathcal{O})$*, then* $o(X) \leq w(x)^{\aleph_0}$*.*

Proof. Fix a base $\mathcal{B}$ of cardinality $w(X)$. Then for any open set U, there exists some $\mathcal{U} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{U}$. Finally, by Corollary 8.30 and Observation 1.28 (applied to U), there exists $V \in [\mathcal{U}]^{\leq \aleph_0}$ such that $U = \bigcup \mathcal{V}$. Thus, we have shown that for each open set U, there exists $\mathcal{V} \in [\mathcal{B}]^{\leq \aleph_0}$ such that $U = \bigcup \mathcal{V}$. □