7. 29.12.05

Definition 7.1. A set $A \subset \mathbb{R}$ is of Lebesgue measure 0 if for every $\varepsilon > 0$ there is a family of open intervals $\langle I_n \mid n \in \mathbb{N}\rangle$ that covers A and $\sum_{n\in\mathbb{N}} |I_n| < \varepsilon$.

Definition 7.2. A set $A \subset \mathbb{R}$ is of *strong measure zero* (or SMZ) iff for every sequence $\langle \varepsilon_n | n \in \mathbb{N}\rangle$ there is a family of open intervals $\langle I_n \mid n \in \mathbb{N}\rangle$ that covers A and $|I_n| < \varepsilon_n$ for every $n \in \mathbb{N}$.

Proposition 7.3. SMZ $\Longrightarrow$ *Lebesgue measure zero.*

Proof. Assume the set $A \subset \mathbb{R}$ is of SMZ. Fix $\varepsilon > 0$. Consider the sequence $\langle \varepsilon/2^n | n \in \mathbb{N}\rangle$. Since A is SMZ there is a family of open intervals $\langle I_n | n \in \mathbb{N}\rangle$ that covers A and $|I_n| < \varepsilon/2^n$ for all $n \in \mathbb{N}$. Since $\sum_{n\in\mathbb{N}} \varepsilon/2^n = \varepsilon$, we get that A is of Lebesgue measure zero. □

Observation 7.4. *If $A \subseteq \mathbb{R}$ is countable, then A is* SMZ*.*

Proof. Suppose $A = \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$ is countable, and $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$ is a sequence of positive reals. For $n \in \mathbb{N}$, let $I_n := (a_n - \frac{\varepsilon_n}{4}, a_n + \frac{\varepsilon_n}{4})$ and observe that $\langle I_n \mid n \in \mathbb{N}\rangle$ works. □

To see that SMZ is much stronger than measure zero, consider for example the Cantor set. We have seen before that it is of measure zero. Is it SMZ? It is obvious that for the sequence $\langle 1/3^n | n \in \mathbb{N}\rangle$, matching open interval cover the Cantor set. Just take $I_1 := (0, 1/3), I_2 = (6/9, 7/9), \ldots$. On the other hand, for the sequence $\langle 1/3^n | n \in \mathbb{N}, n > K \geq 1\rangle$, such family of open intervals that covers the Cantor set can't be obtained (think why?). Therefore it is not SMZ.

Conjecture 7.5 (Borel, 1919)**.** *If $A \subseteq \mathbb{R}$ is* SMZ*, then A is countable.*

Notice that in $\mathbb{R}$, for some open interval $(a, b) \subset \mathbb{R}$, $|(a, b)|$ stands for the length (one dimensional volume) of (a, b), or equivalently, its' diameter. Is it the same in larger metric spaces? Consider for example $\mathbb{R}^2$. The set $[0, 1] \subset \mathbb{R}^2$ is of Lebesgue measure (volume) zero, but the sum of diameters of any open cover consisting with two dimensional "boxes" is not less than 1.[22] The question arises is how to "properly" define SMZ in large metric spaces? Here is the standard way.

Definition 7.6. Suppose $\langle X, d\rangle$ is a metric space.

$A \subseteq X$ is a *strongly null* set iff for any sequence of positive reals, $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$, there is a partition $\{A_n \mid n \in \mathbb{N}\}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$ and $\text{Diam}(A_n) < \varepsilon_n$ for all $n \in \mathbb{N}$.

In the special case of strongly null sets in $\mathbb{R}$, we shall keep call them SMZ.

[22] A box in $\mathbb{R}^2$ is a base set of the product topology, that is a product of open intervals in $\mathbb{R}$

Observation 7.7. *If $\langle X, d\rangle$ is a discrete metric space, then $A \subseteq X$ is strongly null iff A is countable.*

Lemma 7.8. *A uniformly continuous image of a strongly null set is strongly null.*

Proof. Let $\langle X, \rho_X\rangle, \langle Y, \rho_Y\rangle$ be metric spaces where X is strongly null, and let $f : X \to Y$ be uniformly continuous onto Y.

Fix $\varepsilon > 0$. Since f is uniformly continuous, a $\delta > 0$ exists, such that given an open ball $B \subset X$ with $\mathrm{Diam}_{\rho_X}(B) < \delta$ we result with $\mathrm{Diam}_{\rho_Y}(f[B]) < \varepsilon$.

Now, consider some sequence $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$. Implementing the last remark we get a corresponding sequence $\langle \delta_n \mid n \in \mathbb{N}\rangle$. X is strongly null, hence there exist a cover consisting of open balls $\langle B_n \subset X \mid n \in \mathbb{N}\rangle$ where $\mathrm{Diam}_{\rho_X}(B_n) < \delta_n$. For all $n \in \mathbb{N}$ $\mathrm{Diam}_{\rho_Y}(f[B_n]) < \varepsilon_n$.

$$Y = f[X] = f\Big[\bigcup_{n\in\mathbb{N}} B_n\Big] \subseteq \bigcup_{n\in\mathbb{N}} f[B_n] \subseteq \bigcup_{n\in\mathbb{N}} B'_n$$

where $B'_n \subset Y$ are open balls of diameter less than ε_n such that $f[B_n] \subseteq B'_n$. □

Definition 7.9. For a metric space $\langle X, d\rangle$, let $\mathcal{SN}_X := \{A \subseteq X \mid A \text{ is a strongly null set }\}$.

In the special case of $\langle \mathbb{R}, |\cdot|\rangle$, we denote $\mathcal{SN} := \mathcal{SN}_{\mathbb{R}} = \{A \subseteq \mathbb{R} \mid A \text{ is SMZ }\}$.

Proposition 7.10. *For any metric space $\langle X, d\rangle$, $\mathcal{SN}_X$ is a σ-ideal.*[23]

Proof. It is obvious that $\emptyset \in \mathcal{SN}_X$.

Consider some $A \in \mathcal{SN}_X$, and let $B \subset A$. Fix $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$, then since $A \in \mathcal{SN}_X$ there is a cover of A consisting of open set $\langle U_n \mid n \in \mathbb{N}\rangle$ with $\mathrm{Diam}(U_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. Since $B \subseteq \bigcup_{n\in\mathbb{N}} U_n$, we conclude that $B \in \mathcal{SN}_X$.

Finally, to see that $\mathcal{SN}_X$ is σ-additive, assume $\langle A_n \in \mathcal{SN}_X \mid n \in \mathbb{N}\rangle$, and fix $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$. Let $\biguplus_{n\in\mathbb{N}} J_n$ be a partition of $\mathbb{N}$ where J_n is infinite for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. $A_n \in \mathcal{SN}_X$, therefore there is a cover consisting of open sets $\langle U_{n,k} \mid k \in J_n\rangle$ such that $\mathrm{Diam}(U_{n,k}) < \varepsilon_k$ for all $k \in J_n$.

By $\bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} U_{n,k} \supseteq \bigcup_{n\in\mathbb{N}} A_n$, we conclude that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{SN}_X$. □

We have already seen that $\mathcal{SN} \subseteq \mathcal{N}$. We now show a nice connection between SMZ and connectedness.

Claim 7.11. SMZ $\Rightarrow$ 0-*dimensional.*

Proof. Assume $A \in \mathbb{R}$ is SMZ. Recalling Theorem 4.28, it is enough to show that A is totally disconnected. Assume the contrary, that is, it happens that $x \in I \subset A$ where I is connected and $I \setminus \{x\} \neq \emptyset$. I is then an interval which means of positive measure, a contradiction to the fact that A is null (Proposition 7.3). □

[23] A σ-ideal is an ideal closed to countable unions, i.e., $\mathsf{add}(\mathcal{SN}_X) \geq \aleph_1$.

We now reveal the combinatorics of SMZ.

Definition 7.12 (Rothberger)**.** For $k \in \mathbb{N}$, a space $\langle X, O\rangle$ satisfies *Rothberger's property* or $S_k(\mathcal{O}, \mathcal{O})$ iff for any family of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^k \mid n \in \mathbb{N}\rangle$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X.

Observation 7.13. *For a topological space $\langle X, O\rangle$, TFAE:*

(a) $X \models S_1(\mathcal{O}, \mathcal{O})$.
(b) $X \models S_k(\mathcal{O}, \mathcal{O})$ *for some* $k \in \mathbb{N}$.
(c) $X \models S_f(\mathcal{O}, \mathcal{O})$ *for some* $f \in \mathbb{N}^{\mathbb{N}}$*, i.e., for any family of open covers of* X*,* $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$*, there exists a family* $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{f(n)} \mid n \in \mathbb{N}\rangle$*, such that* $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$

Proof. To see (c)⇒(a), fix $f \in \mathbb{N}^{\mathbb{N}}$ such that $X \models S_f(\mathcal{O}, \mathcal{O})$.

Pick an arbitrary partition $\langle A_n \in [\mathbb{N}]^{f(n)} \mid n \in \mathbb{N}\rangle$ with $\biguplus_{n\in\mathbb{N}} A_n = \mathbb{N}$.

For all $n \in \mathbb{N}$, let $\mathcal{V}_n := \{\bigcap \mathrm{Im}(g) \mid g \in \prod_{m\in A_n} \mathcal{U}_m\}$.[24] Evidently, each $\mathcal{V}_n$ covers X.

Applying $S_f(\mathcal{O}, \mathcal{O})$ to $\langle \mathcal{V}_n \mid n \in \mathbb{N}\rangle$, we get a family $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{f(n)} \mid n \in \mathbb{N}\rangle$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Pick $\langle \mathcal{G}_n \in [\prod_{m\in A_n} \mathcal{U}_m]^{f(n)} \mid n \in \mathbb{N}\rangle$ such that $\mathcal{F}_n = \{\bigcap \mathrm{Im}(g) \mid g \in \mathcal{G}_n\}$ for all $n \in \mathbb{N}$. By $|\mathcal{G}_n| = f(n) = |A_n|$, we may enumerate $\mathcal{G}_n = \{g_i \in \prod_{m\in A_n} \mathcal{U}_m \mid i \in A_n\}$.

In this notation, we get that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n = \{\bigcap \mathrm{Im}(g_i) \mid i \in \mathbb{N}\}$.

Finally, since $\bigcap \mathrm{Im}\, g_i \subseteq g_i(i) \in \mathcal{U}_i$ for all $i \in \mathbb{N}$, we get that $\langle g_n(n) \mid n \in \mathbb{N}\rangle$ exemplifies $X \models S_1(\mathcal{O}, \mathcal{O})$. □

Observation 7.14. *Assume $\langle X, d\rangle$ is a metric space.*

For all $Y \subseteq X$, $Y \models S_1(\mathcal{O}, \mathcal{O})$ implies that Y is strongly null.

Proof. Consider a family of positive reals $\langle \varepsilon_n \in \mathbb{R} \mid n \in \mathbb{N}\rangle$.

Fix a basis $\mathcal{B}$ for $\langle X, d\rangle$, and put $\mathcal{U}_n := \{U \in \mathcal{B} \mid \mathrm{Diam}(U) < \varepsilon_n\}$ for each $n \in \mathbb{N}$. By applying $S_1(\mathcal{O}, \mathcal{O})$ of Y to $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, we obtain a family $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $Y \subseteq \bigcup_{n\in\mathbb{N}} U_n$, and obviously, $\mathrm{Diam}(U_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. □

Corollary 7.15. *A Luzin set is an uncountable strongly null set.*

In particular, Borel's conjecture 7.5 is consistently false.

Proof. By Claim 3.25 and the preceding observation. □

Our reader might conjecture that Observation 7.14 can be improved and $S_1(\mathcal{O}, \mathcal{O})$ is actually equivalent to strongly null. However, this is not the case. By Proposition 7.10, strongly null is an hereditary property, whereas we have the following.

Observation 7.16. $S_{fin}(\mathcal{O}, \mathcal{O})$ *is non-hereditary.*

[24] $g \in \prod_{m\in A_n} \mathcal{U}_m$ means that $\mathrm{dom}(g) = A_n$ and $g(m) \in \mathcal{U}_m$ for all $m \in A_n$.

Proof. $\mathbb{R}$ is σ-compact, thus by Lemma 1.29, $\mathbb{R} \models S_{fin}(\mathcal{O},\mathcal{O})$. However, by Theorem 2.29 $\mathbb{R}\setminus\mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. It follows from Theorem 4.10 that $\mathbb{R}\setminus\mathbb{Q} \not\models S_{fin}(\mathcal{O},\mathcal{O})$. □

Observation 7.17. *$S_1(\mathcal{O},\mathcal{O})$ is consistently non-hereditary.*

Proof. Assume $\mathfrak{d} = \aleph_1$. Consider $M := \psi[D] \cup (\mathbb{Q}\cap[0,1])$ of Theorem 4.20. Then M is $\aleph_1$-concentrated on the countable set $\mathbb{Q}\cap[0,1]$, thus, $M \models S_1(\mathcal{O},\mathcal{O})$. However, By Theorem 4.10, $\psi[D]$ does not even satisfy $S_{fin}(\mathcal{O},\mathcal{O})$ (since $\psi^{-1}[\psi[D]] = D$ is dominating), not to mention $S_1(\mathcal{O},\mathcal{O})$. □

It follows from Observation 4.8 that $(\mathfrak{d} = \aleph_1) \implies (\mathfrak{d} = \mathsf{cov}(\mathcal{M}))$. It will soon be clear that it suffices to assume $\mathfrak{d} = \mathsf{cov}(\mathcal{M})$ to conclude that $M \models S_1(\mathcal{O},\mathcal{O})$.

Definition 7.18. A set $X \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be *guessed* by $g \in \mathbb{N}^{\mathbb{N}}$ iff $\{n\in\mathbb{N} \mid f(n) = g(n)\}$ is infinite for all $f \in X$.

Theorem 7.19. *Suppose $X \subseteq \mathbb{N}^{\mathbb{N}}$. If $|X| < \mathsf{cov}(\mathcal{M})$, then X can be guessed.*

Proof. For all $f \in X$ and $k \in \mathbb{N}$, it is obvious that:

$$A_{f,k} := \big\{g \in \mathbb{N}^{\mathbb{N}} \mid \exists n \in \mathbb{N}\big((n > k) \wedge g(n) = f(n)\big)\big\}$$

is dense open. Clearly, any $g \in \bigcap_{f\in X}\bigcap_{k\in\mathbb{N}} A_{f,k}$ will do, so assume towards a contradiction that $\bigcap_{k\in\mathbb{N}}\bigcap_{f\in X} A_{f,k} = \emptyset$. It follows that $\mathbb{N}^{\mathbb{N}} = \bigcup_{k\in\mathbb{N}}\bigcup_{f\in X} B_{f,k}$, where $B_{f,k} := \mathbb{N}^{\mathbb{N}}\setminus A_{f,k}$ are nowhere dense sets. Identifying $\mathbb{N}^{\mathbb{N}}$ with $\mathbb{R}\setminus\mathbb{Q}$, we get that:

$$\mathbb{R} = \bigcup_{k\in\mathbb{N}}\bigcup_{f\in X} B_{f,k} \cup \bigcup_{q\in\mathbb{Q}}\{q\}$$

is the union of $|X|$ nowhere dense sets, contradicting $|X| < \mathsf{cov}(\mathcal{M})$. □

Theorem 7.20. *If $\langle X, O\rangle$ is a topological space and $X \models S_1(\mathcal{O},\mathcal{O})$, then any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ can be guessed.*

Proof. This essentially is the same proof as of Theorem 4.11. Assume some $X \subseteq \mathbb{N}^{\mathbb{N}}$ with $X \models S_1(\mathcal{O},\mathcal{O})$. Fix $m \in \mathbb{N}$. Put $\mathcal{U}_m := \{(m,k)^{\uparrow} \mid k \in \mathbb{N}\}$ where $(m,k)^{\uparrow} := \{f \in \mathbb{N}^{\mathbb{N}} \mid f(m) = k\}$ for all $k \in \mathbb{N}$. Evidently, $\mathcal{U}_m$ is an open cover of X. Fix a bijection $\psi : \mathbb{N}\times\mathbb{N} \leftrightarrow \mathbb{N}$.

Fix $i \in \mathbb{N}$. Since $X \models S_1(\mathcal{O},\mathcal{O})$ and $\langle \mathcal{U}_{\psi(i,n)} \mid n \in \mathbb{N}\rangle$ is a countable family of open covers of X, there exists $g_i : \psi[\{i\}\times\mathbb{N}] \to \mathbb{N}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} \Big(\psi(i,n), g\big(\psi(i,n)\big)\Big)^{\uparrow}$.

Let $g : \mathbb{N} \to \mathbb{N}$ be $g := \bigcup_{n\in\mathbb{N}} g_n$. It is evident that g guesses X. □

Theorem 7.21 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ can be guessed, then $X \models S_1(\mathcal{O},\mathcal{O})$.

Proof. Assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N} \rangle$. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. We may also assume for all $n \in \mathbb{N}$ that members of $\mathcal{U}_n$ are mutually-disjoint, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^\mathbb{N}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^\mathbb{N}$ be the map $x \mapsto f_x$.

Since ψ is continuous, we may pick $g \in \mathbb{N}^\mathbb{N}$ that guesses $\psi[X]$.

For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. To see that $\langle U_n \mid n \in \mathbb{N} \rangle$ covers X. Notice that for each $x \in X$, there exists some $n \in \mathbb{N}$ such that $f_x(n) = g(n)$, i.e., $x \in U_n^{g(n)} = U_n$. □

Corollary 7.22. *For all $X \subseteq \mathbb{R}$, TFAE:*

- $X \models S_1(\mathcal{O}, \mathcal{O})$.
- *Any continuous image of X into $\mathbb{N}^\mathbb{N}$ can be guessed.*

Proof. By theorems 7.20,7.21 and 7.11. □

Corollary 7.23. $X \models S_1(\mathcal{O}, \mathcal{O})$ *for all* $X \in [\mathbb{R}]^{<\mathsf{cov}(\mathcal{M})}$.

Corollary 7.24. *If $X \subseteq \mathbb{R}$ is $\mathsf{cov}(\mathcal{M})$-concentrated on one of its countable subsets, then $X \models S_1(\mathcal{O}, \mathcal{O})$.*

Corollary 7.25. *If $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$, then M of Theorem 4.20 satisfies $S_1(\mathcal{O}, \mathcal{O})$.*

To complete the picture, we mention the following important result.

Theorem 7.26 (Laver)**.** *Borel's conjecture 7.5 is consistent.*

It follows from Corollary 7.15 and the preceding that Borel's Conjecture is independent of the usual axioms of mathematics (ZFC).

Definition 7.27. A set $X \subseteq \mathbb{N}^\mathbb{N}$ is *strongly unbounded* iff for all $f \in \mathbb{N}^\mathbb{N}$, $|X \cap \underline{\{f\}}| < |X|$.

Intuitively, strongly unbounded sets needs to be "fat" enough to be unbounded, but "slim" enough to be strongly unbounded. For instance, $\mathbb{N}^\mathbb{N}$ is indeed unbounded, but it is too "fat" to be strongly-unbounded, recalling Observation 4.9.

Observation 7.28. *There exists strongly unbounded families of cardinality $\mathfrak{b}$ and $\mathfrak{d}$.*

Proof. By Lemmas 1.11 and 1.12. □

Observation 7.29. *Suppose $X \subseteq \mathbb{N}^\mathbb{N}$ is a set such that :*

- $\mathrm{cf}\,|X| > \aleph_0$,
- *For all $f \in \mathbb{N}^\mathbb{N}$, $|\{g \in X \mid g \leq f\}| < |X|$.*

then, X is strongly unbounded.

Proof. Because $\{g \in \mathbb{N}^{\mathbb{N}} \mid g \leq^* f\}$ can be obtained as the following countable union:

$$\bigcup \big\{\{g \in \mathbb{N}^{\mathbb{N}} \mid g \leq f'\} \mid f' \in \mathbb{N}^{\mathbb{N}} \exists N \in \mathbb{N}(\forall n \geq N(f'(n) = f(n))\big\}.$$

□

Let us examine several consequences of Borel's conjecture (BC).

Observation 7.30. *Assuming ZFC+BC, we have:*

(a) $\mathcal{SN} = [\mathbb{R}]^{\leq\omega}$.
(b) $X \subseteq \mathbb{R}$ *satisfies* $S_1(\mathcal{O}, \mathcal{O})$ *iff* X *is countable.*
(c) *Any (continuous) image of* SMZ *is* SMZ*.*
(d) *There is no Luzin set.*
(e) *For any uncountable cardinal* $\kappa \leq \mathsf{cov}(\mathcal{M})$*, there is no strongly unbounded family* $X \in [\mathbb{N}^{\mathbb{N}}]^{\kappa}$.
(g) $\mathsf{cov}(\mathcal{M}) < \min\{\mathsf{cof}(\mathcal{M}), \mathfrak{b}\}$. *In particular* $\mathfrak{b} > \aleph_1$ *and* $\neg$*CH.*

Proof. (a) is equivalent to BC. (b) follows from Observation 7.14. (c) follows from the fact that an image of a countable set is countable. (d) follows from Corollary 7.15.

(e) If $X \subseteq \mathbb{N}^{\mathbb{N}}$ is strongly-unbounded and $\psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is an homeomorphism, then $\psi[X] \cup (\mathbb{Q} \cap [0,1])$ is $|X|$-concentrated at $\mathbb{Q} \cap [0,1]$. Now if X is strongly-unbounded and $|X| \leq \mathsf{cov}(\mathcal{M})$, then by Corollary 7.24 and Observation 7.14, $\psi[X] \cup (\mathbb{Q} \cap [0,1])$ is SMZ.

(f) If $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, then we may apply Theorem 3.7 to obtain a subset of $\mathbb{R}$ which is $\mathsf{cov}(\mathcal{M})$-concentrated at any of its countable dense subsets. Now apply Corollary 7.24 and Observation 7.14.

If $\mathsf{cov}(\mathcal{M}) = \mathfrak{b}$, then Observation 7.28 would have contradict the preceding item.

Finally, by $\mathfrak{b} > \mathsf{cov}(\mathcal{M})$, we have:

$$\mathfrak{c} \geq \mathfrak{b} > \mathsf{cov}(\mathcal{M}) \geq \mathsf{add}(\mathcal{M}) \geq \aleph_1.$$

□

Question 7.31. Is it always true that the continuous image of SMZ is SMZ?

We had already seen that, consistently, SMZ and $S_1(\mathcal{O}, \mathcal{O})$ are different properties, e.g., assuming CH, $S_1(\mathcal{O}, \mathcal{O})$ is non-hereditary, while $\mathcal{SN}$ is an ideal. To answer our question (negatively), we introduce the following theorem:

Theorem 7.32 (Fremlin-Miller)**.** *For* $X \subseteq \mathbb{R}$*, TFAE:*

(a) $X \models S_1(\mathcal{O}, \mathcal{O})$.
(b) *Any continuous image of* X *into* $\mathbb{R}$ *is strongly null.*

Corollary 7.33. *Assuming* CH*, there exists a* SMZ *set* $X \subseteq \mathbb{R}$ *and a continuous function* $f : X \to \mathbb{R}$*, such that* $f[X]$ *is not* SMZ*.*

It happens that the converse of Theorem 7.19 is also true.

Fact 7.34. *There exists* $X \in [\mathbb{N}^{\mathbb{N}}]^{\mathsf{cov}(\mathcal{M})}$ *that cannot be guessed.*

In particular, the minimal cardinality of $A \subseteq \mathbb{R}$ *with* $A \not\models S_1(\mathcal{O}, \mathcal{O})$ *is* $\mathsf{cov}(\mathcal{M})$*.*

Together with Observation 7.30, we obtain that assuming ZFC+BC: $\mathsf{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{b}$.