5. 08.12.05

Proposition 5.1. $\mathbb{N}^{\mathbb{N}}$ *has a countable base consisting of clopen sets.*

Proof. $\left\{\{(n_1,\dots,n_k)\}\times\mathbb{N}^{\mathbb{N}}\,\middle|\, n_1,\dots,n_k,k\in\mathbb{N}\right\}$ is a countable base for $\mathbb{N}^{\mathbb{N}}$ (Recall Example 2.14). The complement of a base set $\{(n_1,\dots,n_k)\}\times\mathbb{N}^{\mathbb{N}}$, is equal to the union of all sets of the form $\{(m_1,\dots,m_k)\}\times\mathbb{N}^{\mathbb{N}}$ where exists $i\le k$ such that $m_i\neq n_i$. This is a union of open sets, hence open. Therefore $\{(m_1,\dots,m_k)\}\times\mathbb{N}^{\mathbb{N}}$ is also closed. □

$\mathcal{B} := \left\{(a,b)\cap(\mathbb{R}\setminus\mathbb{Q})\mid a,b\in\mathbb{Q}\right\} = \left\{[a,b]\cap(\mathbb{R}\setminus\mathbb{Q})\mid a,b\in\mathbb{Q}\right\}$ is a countable family of clopen sets, admitting a base to $\mathbb{R}\backslash\mathbb{Q}$. Applying 2.29, we have another proof to Proposition 5.1.

Definition 5.2. Whenever $\langle X,O\rangle$ is a topological space whose topology O is a metric topology[14] (generated by some metric ρ), we say that $\langle X,O\rangle$ is a *metrizable* topological space.

In this case we can say that the metric is *compatible* with the topology.

Lemma 5.3. *Every metric ρ on a set X is equivalent to a bounded metric.*[15]

Proof. There are two standard ways of replacing ρ by a bounded metric: define new functions ρ_1 and ρ_2 on $X\times X$ by

$$\rho_1(x,y) := \min\{1,\rho(x,y)\}$$

$$\rho_2(x,y) := \frac{\rho(x,y)}{1+\rho(x,y)}$$

We will show that ρ_1 is indeed a metric on X, generating the same topology as ρ does. The reader may verify the same for ρ_2.

ρ_1 is a metric:

- $\rho_1(x,y)=\min\{1,\rho(x,y)\}\ge 0$ since $\rho(x,y)\ge 0$.
- $\rho_1(x,y)=0$ iff $\rho(x,y)=0$ and this occur iff $x=y$.
- $\rho_1(x,z)=\min\{1,\rho(x,z)\}\le\min\{1,\rho(x,y)+\rho(y,z)\}\le\min\{1,\rho(x,y)\}+\min\{1,\rho(y,z)\} = \rho_1(x,y)+\rho_1(y,z)$

ρ_1 generates the same topology as ρ does: on one hand, for some $d>0$, $\mathsf{B}^{\rho_1}_d(x)\supseteq\mathsf{B}^{\rho}_{\min\{1,d\}}(x)$. On the other hand, for some $d<1$, $\mathsf{B}^{\rho_1}_d(x)=\mathsf{B}^{\rho}_d(x)$ (where $\mathsf{B}^{\rho_1}_d(x)$ for example is the set $\{y\in X\mid\rho_1(x,y)<d\}$). □

Theorem 5.4. *A product space $\prod_{n\in\mathbb{N}}X_n$ is metrizable iff each space X_n is metrizable.*

[14] Open balls generated by any metric is always a topology base.

[15] Two metrics on a set are equivalent if they generate the same topology.

Proof. ($\Rightarrow$) Each X_n is homeomorphic to a subspace of the product space, hence metrizable.

($\Leftarrow$) Let $\langle\langle X_n, \rho_n\rangle \mid n \in \mathbb{N}\rangle$ be a family of metric spaces with $\operatorname{Im}(\rho_n) \subseteq [0,1]$ for all $n \in \mathbb{N}$.

Define ρ on $X := \prod X_i$ as follows: for $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$

$$\rho(x,y) := \sum_{i\in\mathbb{N}} \frac{\rho_i(x_i, y_i)}{2^i}.$$

It is easily verified to be a metric. We will show that it gives the product topology in X.

Pick $x = (x_1, x_2, ...) \in X$ and assume $B_x \subseteq X$ is an open set containing x. We may assume that B_x is the is of the following form:

$$B_x = \mathsf{B}_{\varepsilon_1}(x_1) \times \cdots \times \mathsf{B}_{\varepsilon_n}(x_n) \times \prod_{k>n} X_k.$$

where $\mathsf{B}_{\varepsilon_i}(x_i) = \{y \in X_i \mid \rho_i(y, x_i) < \varepsilon_i\}$ for all relevant i.

Put $\varepsilon := \min\left(\frac{\varepsilon_1}{2}, ..., \frac{\varepsilon_n}{2^n}\right)$. Now, if $\rho(x,y) < \varepsilon$, then $\rho_i(x_i, y_i) < \varepsilon_i$ for all $i \in \mathbb{N}$, so apparently $\mathsf{B}_\varepsilon(x) \subset B_x$. Thus the product topology on X is weaker that the topology induced by ρ. On the other hand, given $\varepsilon > 0$, we can choose N large enough that $\sum_{i\geq N+1} \frac{1}{2^i} < \varepsilon/2$. Then it is easily verified that $\mathsf{B}_{\frac{\varepsilon}{2N}}(x_1) \times \cdots \times \mathsf{B}_{\frac{\varepsilon}{2N}}(x_N) \times \prod_{k>N} X_k \subset \mathsf{B}_\varepsilon(x)$, hence, the topology induced by ρ is weaker the the product topology. □

Corollary 5.5. $\mathbb{N}^\mathbb{N}$ *is a metric-space.*

Proposition 5.6. $\mathbb{N}^\mathbb{N}$ *is a complete metric space.*

Proof. For $f, g \in \mathbb{N}^\mathbb{N}$, denote by $N(f,g) := \min\{n \in \mathbb{N} \mid f(n) \neq g(n)\}$. Now, define $\rho(f,g) := \frac{1}{N(f,g)}$. As in the proof of Theorem 5.4, ρ is a metric that is compatible with the usual product topology of $\mathbb{N}^\mathbb{N}$.

Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For $K \in \mathbb{N}$, there exists $N_K \in \mathbb{N}$ such that $d(f_l, f_m) < 1/K$ for all $l, m \geq N_K$. By definition of ρ this means that $f_l(n) = f_m(n)$ for all $l, m \geq N_k$ and $n \leq K$.

Define $f \in \mathbb{N}^\mathbb{N}$ as follows: for every $n \in \mathbb{N}$ define $f(n) := f_{N_n}(n)$. Obviously, $d(f_n, f) \to 0$ as $n \to \infty$, concluding that $\mathbb{N}^\mathbb{N}$ is complete. □

Corollary 5.7. $\mathbb{N}^\mathbb{N}$ *is a Baire space.*

Notice that if a space is locally compact, then it is also a Baire space, this is essentially due to Lemma 3.13 and Theorem 3.16.

Now, Since $\mathbb{R}$ is locally compact, and $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$,[16] we know that $\mathbb{N}^\mathbb{N}$ is also locally compact[17]. This gives another proof for the preceding Corollary.

[16]**homeomorphic**, not isometric.

[17]Recall Corollary 4.27.

Definition 5.8. Suppose $\langle X, O\rangle$ is a topological space. A family of open sets $\mathcal{U} \subseteq O$ is a *γ-cover* iff $\mathcal{U}$ is infinite, and for all $x \in X$, $\{U \in \mathcal{U} \mid x \notin U\}$ is finite.

Thus, for instance, $\{(-n, n) \mid n \in \mathbb{N}\}$ is a γ-cover of $\mathbb{R}$.

Observation 5.9. *If $\mathcal{U}$ is a γ-cover of some space $\langle X, O\rangle$, then any infinite subset $\mathcal{V} \subseteq \mathcal{U}$ is a γ-cover.*

In particular, any γ-cover contains a countable γ-cover.

Observation 5.10. *Suppose $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is an open cover of some space $\langle X, O\rangle$, then either $\mathcal{U}$ contains a finite subcover, or that $\mathcal{V} := \{\bigcup_{m \leq n} U_n \mid n \in \mathbb{N}\}$ is a γ-cover of X.*

Proof. If $\mathcal{U}$ does not contain a finite subcover, then $\mathcal{V}$ is infinite, and is clearly a γ-cover. □

Definition 5.11. For a topological space $\langle X, O\rangle$ denote $\mathcal{O} := \{\mathcal{U} \subseteq O \mid \mathcal{U}$ is an open cover of $X\}$ and $\Gamma := \{\mathcal{V} \subseteq O \mid \mathcal{V}$ is an open γ-cover of $X\}$.

Definition 5.12 (Hurewicz)**.** A space $\langle X, O\rangle$ satisfies *Hurewicz's property* or $U_{fin}(\mathcal{O}, \Gamma)$ iff for any sequence of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$, such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ forms a γ-cover of X.

Observation 5.13. *$U_{fin}(\mathcal{O}, \Gamma)$ is a topological property and there also exists an analogue of Observation 1.31 for $U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. Essentially the same proofs of 2.1 and 1.31. □

To compare the definition of $U_{fin}(\mathcal{O}, \Gamma)$ with $S_{fin}(\mathcal{O}, \mathcal{O})$ (Definition 1.26), it is evident that the left hand side set ($\mathcal{O}$ in both cases) is the requirement that $\mathcal{U}_n \in \mathcal{O}$ for all $n \in \mathbb{N}$.

Now, for the right hand side, in the first case we need to generate a γ-cover, that is, a member of Γ, while, on the other, we need to generate an open cover, that is, a member of $\mathcal{O}$. The generation is always based at some finite sets $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$, where S "says" that the object is obtained by taking $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, and U says that the object is obtained by considering $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$.

Observation 5.14. *$X \models S_{fin}(\mathcal{O}, \Gamma)$ implies that any open cover of X contains a γ-cover.*

Consequently, no topological space X satisfies $S_{fin}(\mathcal{O}, \Gamma)$.

Proof. For an open cover $\mathcal{U}$, consider $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$ where $\mathcal{U}_n := \mathcal{U}$ for all $n \in \mathbb{N}$. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subseteq \mathcal{U}$ is a γ-cover.

To see the second assertion, take $\mathcal{U} := \{X\}$. □

Observation 5.15. *$U_{fin}(\mathcal{O}, \Gamma) \Rightarrow S_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. We assume a topological space $\langle X, O\rangle$ and $\langle U_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \Gamma$.

We claim that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Indeed, since $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ covers X, we have:

$$X \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{F}_n = \bigcup \bigcup_{n\in\mathbb{N}} \mathcal{F}_n.$$

□

We can now obtain the result of Lemma 1.29 as an application of the preceding together with the following.

Lemma 5.16. *If $\langle X, O\rangle$ is a σ-compact topological space, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. Suppose $\langle K_n \mid n \in \mathbb{N}\rangle$ is an increasing sequence of compact subspaces of X, whose union is X, and $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover of X. By compactness of each factor, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $K_n \subseteq \bigcup \mathcal{F}_n$ for all $n \in \mathbb{N}$. Finally, since $\langle K_n \mid n \in \mathbb{N}\rangle \nearrow X$, we conclude that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is a γ-cover of X (it is infinite because each $\mathcal{U}_n$ does not contain a finite subcover). □

Conjecture 5.17 (Hurewicz)**.** *$U_{fin}(\mathcal{O}, \Gamma)$ is equivalent to σ-compactness.*

The reader might want to compare the above with Conjecture 1.30. To continue the research, we need the following reduction theorem, an analogue of Theorem 4.10.

Theorem 5.18 (Hurewicz)**.** *For all $X \subseteq \mathbb{R}$, TFAE:*

- *$X \models U_{fin}(\mathcal{O}, \Gamma)$.*
- *Any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.*

Proof. We omit the proof. Instead, we prove the following two propositions. □

Theorem 5.19. *If $\langle X, O\rangle$ is a topological space and $X \models U_{fin}(\mathcal{O}, \Gamma)$, then any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.*

Proof. By Observation 5.13, we may assume that $X \subseteq \mathbb{N}^{\mathbb{N}}$ and $X \models U_{fin}(\mathcal{O}, \Gamma)$. Fix $n \in \mathbb{N}$. Put $\mathcal{U}_n := \{(n,k)^{\uparrow} \mid k \in \mathbb{N}\}$. Evidently, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle \in \mathcal{O}$, so let $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ witness $U_{fin}(\mathcal{O}, \Gamma)$. Define $g : \mathbb{N} \to \mathbb{N}$. For $n \in \mathbb{N}$, let $g(n) := 1 + \max\{k \in \mathbb{N} \mid (n,k)^{\uparrow} \in \mathcal{F}_n\}$.

To see that $X \subseteq \underline{\{g\}}$, we pick $f \in X$ and show that $f \leq^* g$.

Since $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \Gamma$, there exists some $N \in \mathbb{N}$, such that $f \in \bigcup \mathcal{F}_n$ for all $n \geq N$, that is, $f(n) \leq g(n)$ for all $n \geq N$, and we are done. □

Theorem 5.20 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-boudned, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.

Proof. By Observation 5.13, we assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. We may also assume that members of $\mathcal{U}_n$ are mutually-disjoint for all $n \in \mathbb{N}$, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^\mathbb{N}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^\mathbb{N}$ be the map $x \mapsto f_x$.

Since ψ is continuous, we may pick $g \in \mathbb{N}^\mathbb{N}$ witnessing that $\psi[X]$ is $\leq^*$-boudned. For all $n \in \mathbb{N}$, put $\mathcal{F}_n := \{U_n^1, .., U_n^{g(n)}\}$. To see that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is a γ-cover, fix $x \in X$. By definition of g, there exists some $N \in \mathbb{N}$ such that $f_x(n) \leq g(n)$ for all $n \geq N$, and hence, $x \in \bigcup \mathcal{F}_n$ for all $n > N$. As usual, $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is infinite because each $\mathcal{U}_n$ does not contain a finite subcover. □

Corollary 5.21. *If $X \in [\mathbb{R}]^{<\mathfrak{b}}$, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. By ($\Leftarrow$) of Theorem 5.18. □

The next is similar to Corollary 4.19.

Corollary 5.22. *Any uncountable $X \in [\mathbb{R}]^{<\mathfrak{b}}$ is a counter-example to Hurewicz's conjecture. In particular, Hurewicz's conjecture 5.17 is consistently false.*

Proof. Suppose $\mathfrak{b} > \aleph_1$ (this assumption is consistent) and $X \in [\mathbb{R}]^{<\mathfrak{b}}$ is uncountable. If X was σ-compact, then by Lemma 3.23, it had contained a perfect subset and by Lemma 3.26, X had to contained a set of size $\mathfrak{c}$, contradicting $|X| < \mathfrak{b} \leq \mathfrak{c}$. □

Observation 5.23. *Consistently, there exists $X \subseteq \mathbb{N}^\mathbb{N}$ such that:*

(a) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$,
(b) $X \not\models U_{fin}(\mathcal{O}, \Gamma)$ *(and in particular, X is not σ-compact).*

Thus, consistently: Menger's conjecture 1.30 has a counter-example already inside $\mathbb{N}^\mathbb{N}$, and Observation 5.14 cannot be improved.

Proof. Put $\mathcal{J} := \{Y \subseteq \mathbb{N}^\mathbb{N} \mid Y \text{ is meager }\}$. By Corollary 5.7, $\mathcal{J}$ is a proper ideal. Assume $c = \aleph_1$ (this is consistent), or even the weaker assumption that $\mathsf{cov}(\mathcal{J}) = \mathsf{cof}(\mathcal{J})$.

For $X := A$, the set given by Theorem 3.7 by taking $I = \mathcal{J}$, the same argument of the proof of Claim 3.25 shows that A is $\mathsf{cov}(\mathcal{J})$-concentrated on one of its countable (dense) subsets. Now, since $\mathcal{I}_\mathfrak{b} \subseteq \mathcal{J}$, we have $\mathsf{cov}(\mathcal{J}) \leq \mathsf{cov}(\mathcal{I}_\mathfrak{b}) = \mathfrak{d}$. Thus, we noticed that there exists $D \in [X]^{\aleph_0}$ such that X is $\mathfrak{d}$-concentrated at D, and hence $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

To see that $X \not\models U_{fin}(\mathcal{O}, \Gamma)$, notice that $X \notin \mathcal{J}$ implies $X \notin \mathcal{I}_\mathfrak{b}$ and recall Theorem 5.19. □

With the notation the above proof, it is very interesting to notice that even if $\mathfrak{c} = \aleph_1$ (and hence $\mathfrak{b} = \mathfrak{d}$), then still, somehow, the diagonalization process of Theorem 3.7 will generate $X \subseteq [\mathbb{N}^\mathbb{N}]$ (of cardinality $\mathfrak{b} = \mathfrak{d}$), which is $\leq^*$-unbounded, but not $\leq^*$-dominating.

Definition 5.24. A function $f : X \to Y$ between two topological spaces is a *Borel function* iff the preimage of an open set (in Y) is Borel (in X).

Thus, Borel function is a weakening of continuous function.

Theorem 5.25 (Kuratowski)**.** *If $S \subseteq [0,1] \subseteq \mathbb{R}$ and $f : S \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then there exists an extension $g : [0,1] \to \mathbb{N}^{\mathbb{N}}$ such that g is a Borel function and $g \restriction S = f$.*

Theorem 5.26 (Luzin)**.** *If $f : [0,1] \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then for all $n \in \mathbb{N}$, there exists some closed subset $C_n \subseteq [0,1]$ such that $f \restriction C_n$ is continuous and C_n is of Lebesgue measure $\geq 1 - \frac{1}{n}$.*

The next is similar to Theorem 3.24.

Theorem 5.27. *A Sierpinski subset of $[0,1]$ is a counter-example to Hurewicz's conjecture. In particular, Hurewicz's conjecture 5.17 is consistently false.*

Proof. Let $S \subseteq [0,1]$ be a Sierpinski set. The consistency of existence of such set follows, e.g., from $\mathfrak{c} = \aleph_1$ and the proof of Corollary 3.8 applied to $\mathcal{N}_{[0,1]}$ instead of to $\mathcal{N}$.

Claim 5.28. *S is not σ-compact.*

Proof. If S was σ-compact, then by Lemma 3.23, it had contain a perfect subset and by Lemma 3.26, S had to contained a null set of size $\mathfrak{c}$, contradicting the fact that S is Sierpinski set. □

We now use Theorem 5.18 to prove that $S \models U_{fin}(\mathcal{O}, \Gamma)$.

Claim 5.29. *Assume $\psi : S \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then $\psi[S] \in \mathcal{I}_{\mathfrak{b}}$.*

Proof. Let $\varphi : [0,1] \to \mathbb{N}^{\mathbb{N}}$ be an extension of ψ given by Theorem 5.25. Let $\langle C_n \subseteq [0,1] \mid n \in \mathbb{N}\rangle$ be like in Theorem 5.26 applied to φ.

For $n \in \mathbb{N}$, the choice of C_n implies that $\varphi[C_n]$ is compact. It follows $\varphi[\bigcup_{n\in\mathbb{N}} C_n] = \bigcup_{n\in\mathbb{N}} \varphi[C_n]$ is σ-compact, and in particular, $\psi[S \cap \bigcup_{n\in\mathbb{N}} C_n] \in \mathcal{I}_{\mathfrak{b}}$. (Recall Lemma 4.7.)

We are left with showing that $\psi[S \setminus \bigcup_{n\in\mathbb{N}} C_n] \in \mathcal{I}_{\mathfrak{b}}$, but this is trivial, because $\bigcup_{n\in\mathbb{N}} C_n$ is of measure 1 and S is a Sierpinski set, so, $S \setminus \bigcup_{n\in\mathbb{N}} C_n$ is countable. □

□

With the notation of the preceding proof, notice that it suffices to assume that S has the property that any intersection of S with a null set is of cardinality $< \mathfrak{b}$, that is, the proof can be carried out flawlessly had we assumed that $S \subseteq [0,1]$ is the set given by Theorem 3.7, whenever $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$.

Definition 5.30. A *compactification* of a space X is a pair (K, h), where K is compact, $h : X \to h(X) \subset K$ is an homeomorphism, and $\overline{h(X)} = K$

We will sometimes simply say that K *is a compactification of* X. In many cases, h will be an inclusion map, so that $X \subset K$.

Definition 5.31. A space $\langle X, O\rangle$ is *locally-compact* iff for all $x \in X$, there exists an open $U \subseteq X$, with $x \in U$ and $\overline{U}$ compact.

Definition 5.32 (Alexandrov compactification)**.** Let $\langle X, O\rangle$ be locally-compact, noncompact Hausdorff space, and $p \notin X$. Define $\langle X^*, O^*\rangle$ by letting $X^* := X \cup \{p\}$ and:

$$O^* := O \cup \big\{\{p\} \cup (X \setminus K) \mid K \subseteq X \text{ is compact}\big\}.$$

We call X^* the *one-point compactification of* X.

Observations:

- Verifying that $\langle X^*, O^*\rangle$ is indeed a topological space is easy.
- X^* is compact. Assume $\{U_s\}_{s\in S}$ is an open cover of X^*.

  It follows that there exist some $s_p \in S$ with $p \in U_{s_p}$, that is, $U_{s_p} = \{p\} \cup (X \setminus K)$ where K is compact in X. Now, $\{U_s\}_{s\in S\setminus s_p}$ is an open cover of K, so there is a finite subcover $\{U_{s_1}, ..., U_{s_n}\}$. We conclude that $\{U_{s_p}, U_{s_1}, ..., U_{s_n}\}$ is a cover of X^*.
- X is open in X^* since X is open in itself.
- X is dense in X^*. Showing that $\{p\}$ is not open will do. Assume that $\{p\}$ is open, meaning $\{p\} = \{p\} \cup (X \setminus X)$ where X is compact. A contradiction, since X is noncompact.
- X^* is Hausdorff. Consider two distinct points x, x' in X^*. If both are in X then we are done since X is Hausdorff. So, assume $x' = p$. X is locally compact, that is, there is an open set $x \in U_x$ such that $\overline{U_x}$ is compact in X, therefore $V_p := \{p\} \cup (X \setminus \overline{U_x})$ is open and $U_x \cap V_p = \emptyset$.

Example 5.33. (1) Consider the real line $\mathbb{R}$, and define $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ with the topology as described. Now, this is actually a space homeomorphic to S^1, the unit sphere in $\mathbb{R}^2$, which is obviously compact.

(2) Actually, the one-point compactification of $\mathbb{R}^n$ is S^n.

Theorem 5.34 (Alexander)**.** *Assume $\langle X, O\rangle$ is a topological space and $\mathcal{S}$ is some subbase for the topology on X.*

If every cover of X with elements of $\mathcal{S}$ has a finite subcover, then X is compact.

Proof. For the sake of the proof, we shall use the following notation:

A collection $\mathcal{U}$ of open sets is $\mathbb{B}$ iff it is not a cover. It is $\mathbb{B}_{fin}$ iff it does not have a finite subcover. We say that a $\mathbb{B}_{fin}$ collection $\mathcal{U}$ is *maximal* iff there exists some open set U such that $\mathcal{U} \cup \{U\}$ is not $\mathbb{B}_{fin}$.

Evidently, $\mathbb{B} \Rightarrow \mathbb{B}_{fin}$, and $\langle X, O\rangle$ is compact iff $\mathbb{B}_{fin} \Rightarrow \mathbb{B}$ for all $\mathcal{U} \subseteq O$.

Lemma 5.35. *Every $\mathbb{B}_{fin}$ collection can be extended to a maximal $\mathbb{B}_{fin}$ collection.*

Proof. Assume $\mathcal{U}_0$ is $\mathbb{B}_{fin}$. Let $\mathcal{A} := \{\mathcal{U} \mid \mathcal{U}_0 \subseteq \mathcal{U} \subseteq O \text{ is } \mathbb{B}_{fin}\}$. $\mathcal{A}$ is clearly non-empty. Naturally, $\langle \mathcal{A}, \subseteq\rangle$ is a partially ordered-set. Now, recall Zorn's Lemma:

Lemma 5.36 (Zorn)**.** *If $\langle P, \leq\rangle$ is a non-empty poset with the property:*
$(\star)$ For all $C \subseteq P$ such that $\langle C, \leq\rangle$ is linearly-ordered, there exists some $y \in P$ such that $x \leq y$ for all $x \in C$.

Then, $\langle P, \leq\rangle$ contains a maximal element m, that is, $m \not< x$ for all $x \in P$.

Clearly, to complete the proof, it suffices to show that the hypothesis of Zorn's Lemma holds. Let $\{\mathcal{U}_i\}_{i\in I} \subseteq \mathcal{A}$ (where I is some index set) be a chain, and define $\mathcal{U} := \bigcup_{i\in I} \mathcal{U}_i$.

Assume now that $\mathcal{U}$ is not $\mathbb{B}_{fin}$, that is, there are $\{U_k\}_{k\leq n} \subset \mathcal{U}$ such that $X = \bigcup_{k\leq n} U_k$. Since there is an increasing sequence $\langle i_k \in I \mid 1 \leq k \leq n\rangle$ such that $U_k \in \mathcal{U}_{i_k}$, we get that $\mathcal{U}_{i_n}$ is $\mathbb{B}_{fin}$. A contradiction. □

So, assume now that $\mathcal{U}$ is a maximal $\mathbb{B}_{fin}$ extension of $\mathcal{U}_0$.

Let J be an arbitrary index set. For all $j \in J$ assume $V_j \notin \mathcal{U}$ is an open set, then there are $\{U_{j_k}\}_{k\leq n_j}$ all in $\mathcal{U}$ such that $V_j \cup \bigcup_{k\leq n_j} U_{k_j} = X$. Therefore $\left(\bigcap_j V_j\right) \cup \left(\bigcup_j \bigcup_{k\leq n_j} U_k\right) = X$. We conclude that there does not exist $U \in \mathcal{U}$ such that $\bigcap_j V_j \subset U$, otherwise $\mathcal{U}$ would not have been $\mathbb{B}_{fin}$. Thus, if $\bigcap_j V_j \subset U$ for some $U \in \mathcal{U}$, then there is $j \in J$ with $V_j \in \mathcal{U}$.

Define $\mathcal{U}' := \mathcal{U} \cap \mathcal{S}$. Let $x \in U \in \mathcal{U}$. There are $\{V_j\}_{j\leq n} \subset \mathcal{S}$ such that $x \in \bigcap_{j\leq n} V_j \subset U$, thus, there is $j \leq n$ such that $V_j \in \mathcal{U}$, therefore $V_j \in \mathcal{U}'$. We conclude that $\bigcup \mathcal{U}' = \bigcup \mathcal{U}$.

Now, assume $X = \bigcup \mathcal{U}$, meaning $X = \bigcup \mathcal{U}'$, but, by the hypothesis, $\mathcal{U}'$ has a subcover for X, therefore so does $\mathcal{U}$, in contradiction to the fact that $\mathcal{U}$ is $\mathbb{B}_{fin}$.

So, $X \neq \bigcup \mathcal{U}$, that is, $\mathcal{U}$ is a $\mathbb{B}$ collection, in particular, $\mathcal{U}_0$ is a $\mathbb{B}$ collection, but we assumed $\mathcal{U}_0$ is $\mathbb{B}_{fin}$.

Since $\mathcal{U}_0$ is an arbitrary $\mathbb{B}_{fin}$ collection, we get that X is compact. □

Theorem 5.37 (Tychonoff)**.** *A nonempty product space is compact iff each factor space (in the product) is compact.*

Proof. ($\Rightarrow$) If the product space is nonempty, then the projection maps are all continuous (see proposition 2.15) and onto, and since the continuous image of a compact space is compact, the result follows.

($\Leftarrow$) Assume $\{X_i\}_{i\in I}$ is a collection of compact spaces, and define $X := \prod_{i\in I} X_i$. Consider the canonical subbase to the topology of X, $\mathcal{S} := \{\pi_i^{-1}[U] \mid i \in I, U \subseteq X_i \text{ is open}\}$.

By Alexander's theorem 5.34, it is sufficient to show that every $\mathbb{B}_{fin}$ collection $\mathcal{U} \subseteq \mathcal{S}$, is also a $\mathbb{B}$ collection, so let us fix such $\mathcal{U}$.

For all $i \in I$, put $\mathcal{U}_i := \{U \subseteq X_i \mid \pi_i^{-1}[U] \in \mathcal{U}\}$.

Lemma 5.38. *For all $i \in I$, $\mathcal{U}_i$ is $\mathbb{B}_{fin}$ in X_i.*

Proof. Assume that $\mathcal{U}_i$ is not $\mathbb{B}_{fin}$ in X_i, then there are $U_1, .., U_n \in \mathcal{U}_i$ such that $\bigcup_{k\le n} U_k = X_i$, hence $X = \pi_i^{-1}[X_i] = \pi_i^{-1}\left[\bigcup_{k\le n} U_k\right] = \bigcup_{k\le n} \pi_i^{-1}[U_k]$. We conclude that $\mathcal{U}$ is $\mathbb{B}_{fin}$. A contradiction. □

Now, since X_i is compact, we must conclude that $\mathcal{U}_i$ is a $\mathbb{B}$ collection (for all $i \in I$), meaning that there exist some $x_i \in X_i \setminus \left(\bigcup \mathcal{U}_i\right)$.

Let $x \in X$ be the only member in X satisfying $\pi_i(x) = x_i$ for all $i \in I$.

Lemma 5.39. *$x \notin \bigcup \mathcal{A}$.*

In particular, $\mathcal{U}$ is a $\mathbb{B}$ collection.

Proof. Assume $x \in \bigcup \mathcal{U}$, then there exists some $U \in \mathcal{U}$ such that $x \in U$, that is, there exists some $i \in I$ and $U_i \subseteq X_I$ such that $x \in U = \pi_i^{-1}[U_i]$

Now, $x \in \pi_i^{-1}[U_i]$ iff $x_i = \pi_i(x) \in U_i$. This is a contradiction to the fact that $x_i \notin \bigcup \mathcal{U}_i$. □

□

It is worth mentioning that Tychonoff's theorem 5.37 is equivalent to the Axiom of Choice (the C of ZFC) which is equivalent to Zorn's Lemma 5.36.

Theorem 5.40 (Scheepers-Just-Miller-Szeptycki)**.** *Hurewicz's conjecture 5.17 is false.*

We omit the original proof. Instead, in the next lecture we shall introduce an alternative, simpler, proof due to Bartoszyński and Tsaban.