4. 02.12.05

Observation 4.1. *For any open $U \subseteq \mathbb{N}^{\mathbb{N}}$, $|U| = \mathfrak{c}$ and $\underline{U} = \mathbb{N}^{\mathbb{N}}$.*

Observation 4.2. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $\{f \in \mathbb{N}^{\mathbb{N}} \mid g \leq^* f\}$ is dense in $\mathbb{N}^{\mathbb{N}}$.*

Lemma 4.3. *Suppose $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a compact subspace, then there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq g$ for all $f \in Y$.*

Proof. For all $n \in \mathbb{N}$, consider the projection $\pi_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that $\pi_n(f) = f(n)$ for all $f \in \mathbb{N}^{\mathbb{N}}$. By definition of the Baire space, each π_n is continuous and by the hypothesis, Y is compact and it follows that $\pi_n[Y]$ is compact in $\mathbb{N}$. Since any compact subspace of the discrete space $\mathbb{N}$ is finite, we conclude that for all $n \in \mathbb{N}$, there exists some $m_n \in \mathbb{N}$ such that $\pi_n[Y] \subseteq \{1, .., m_m\}$. It other words, the function $g \in \mathbb{N}^{\mathbb{N}}$ defined by $n \mapsto m_n$ has the property that $f \leq g$ for all $f \in Y$ and we are done. □

Observation 4.4. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $D_g := \{f \in \mathbb{N}^{\mathbb{N}} \mid f \leq g\}$ is a closed, nowhere-dense, subspace of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Fix $g \in \mathbb{N}^{\mathbb{N}}$. Assume $h \in \mathbb{N}^{\mathbb{N}} \setminus D_g$. Then there exists some $n \in \mathbb{N}$ such that $h(n) > g(n)$. Then h is in the open set $U = \{f \in \mathbb{N}^{\mathbb{N}} \mid f(n) = h(n)\}$ and $U \subseteq \mathbb{N}^{\mathbb{N}} \setminus D_g$.

To see that $\mathbb{N}^{\mathbb{N}} \setminus D_g$ is dense, we fix a base open set U, and show that $U \cap (\mathbb{N}^{\mathbb{N}} \setminus D_g) \neq \emptyset$. Find $n \in \mathbb{N}$, and $\sigma : \{1, ..n\} \to \mathbb{N}$ such that $U = \sigma^{\uparrow}$. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that $h \restriction \{1, .., n\} = \sigma$ and $h(k) = g(k) + 1$ for all $k > n$. Clearly, $h \in U \setminus D_g$. □

Corollary 4.5. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $E_g := \{f \in \mathbb{N}^{\mathbb{N}} \mid f \leq^* g\}$ is an F_σ meager subspace of $\mathbb{N}^{\mathbb{N}}$.*

Proof. If σ is a finite sequence of natural numbers, we may consider $sw(\sigma, g) \in \mathbb{N}^{\mathbb{N}}$ such that $sw(\sigma, g)(n) = \sigma(n)$ if $n \in \text{dom}(\sigma)$ and $sw(\sigma, g)(n) = g(n)$ otherwise.

Then $E_g = \bigcup\{D_{sw(\sigma,g)} \mid \sigma \text{ is a finite sequence of natural numbers }\}$. □

Definition 4.6. Let $\mathcal{I}_{\mathfrak{b}} := \{X \subseteq \mathbb{N}^{\mathbb{N}} \mid \text{ecf}(X) \leq 1\}$.

It is by the definition of $\mathfrak{b}$ that $\mathcal{I}_{\mathfrak{b}}$ is a non-trivial proper ideal, $\mathsf{add}(\mathcal{I}_{\mathfrak{b}}) = \mathfrak{b}$, and $\mathcal{I}_{\mathfrak{b}}$ contains exactly all sets that are $\leq^*$-bounded in $\mathbb{N}^{\mathbb{N}}$.

Also notice that $\mathcal{I}_{\mathfrak{b}} = \{X \subseteq \mathbb{N}^{\mathbb{N}} \mid \text{ecf}(X) < \mathfrak{b}\}$ and $\mathsf{cov}(\mathcal{I}_{\mathfrak{b}}) = \mathsf{cof}(\mathcal{I}_{\mathfrak{b}}) = \mathfrak{d}$.

Corollary 4.7. *Suppose that $Z \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-compact topological space, then $Z \in \mathcal{I}_{\mathfrak{b}}$, i.e., there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in Z$.*

In particular (since $\aleph_1 \leq \mathfrak{b}$), any σ-compact subspace of $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.

Proof. Let $\langle Z_\alpha \subseteq Z \mid \alpha < \kappa\rangle$ witness $\mathfrak{b}$-compactness of Z (in particular, $\kappa < \mathfrak{b}$). For all $\alpha < \kappa$, Theorem 4.3 implies that $Z_\alpha \in \mathcal{I}_{\mathfrak{b}}$ (and even more, but we don't care). Now, by $\kappa < \mathsf{add}(\mathcal{I}_{\mathfrak{b}})$, $Z = \bigcup_{\alpha<\kappa} Z_\alpha \in \mathcal{I}_{\mathfrak{b}}$ and we are done. □

Observation 4.8. $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$.

Proof. Pick a cofinal subset $D \subseteq [\mathbb{N}^\mathbb{N}]^\mathfrak{d}$ and an homeomorphism $\psi : \mathbb{N}^\mathbb{N} \to \mathbb{R}\backslash\mathbb{Q}$. By Corollary 4.5 and $\{\underline{\{f\}} \mid f \in D\} \subseteq \mathcal{I}_\mathfrak{b}$, we have that $\{\psi[\underline{\{f\}}] \mid f \in D\} \subseteq \mathcal{M}$. Finally, since

$$\mathbb{R} = \psi[\mathbb{N}^\mathbb{N}] \cup \mathbb{Q} = \psi[\bigcup_{f \in D} \underline{\{f\}}] \cup \mathbb{Q} = \bigcup\{\psi[\underline{\{f\}}], \mathbb{Q} \mid f \in D\} =: \bigcup A,$$

and $A \in [\mathcal{M}]^\mathfrak{d}$, we conclude that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. □

Observation 4.9. *There exists $X \in \mathcal{I}_\mathfrak{b}$ with $|X| = \mathfrak{c}$.*

In particular, if $\mathfrak{b} < \mathfrak{c}$, then there exists $X \in \mathcal{I}_\mathfrak{b}$ with $|X| > \mathfrak{b}$.

Proof. Consider $X := \underline{\{f\}}$ where $f : \mathbb{N} \to \{2\}$ is the constant function. □

Theorem 4.10 (Hurewicz)**.** *For all $X \subseteq \mathbb{R}$, TFAE:*

- *$X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*
- *Any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating.*

Proof. We omit the proof. Instead, we prove the following two propositions. □

Theorem 4.11. *If $\langle X, O\rangle$ is a topological space and $X \models S_{fin}(\mathcal{O}, \mathcal{O})$, then any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating.*

Proof. By Lemma 2.1, we may assume that $X \subseteq \mathbb{N}^\mathbb{N}$ and $X \models S_{fin}(\mathcal{O}, \mathcal{O})$. Fix $m \in \mathbb{N}$. Put $\mathcal{U}_m := \{(m, k)^\uparrow \mid k \in \mathbb{N}\}$ where $(m, k)^\uparrow := \{f \in \mathbb{N}^\mathbb{N} \mid f(m) = k\}$ for all $k \in \mathbb{N}$. Evidently, $\mathcal{U}_m$ is an open cover of X (and actually of $\mathbb{N}^\mathbb{N}$). Fix a bijection $\psi : \mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{N}$. Fix $i \in \mathbb{N}$.

Since $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $\langle \mathcal{U}_{\psi(i,n)} \mid n \in \mathbb{N}\rangle$ is a countable family of open covers of X, there exists some $\langle \mathcal{F}_{\psi(i,n)} \in [\mathcal{U}_{\psi(i,n)}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\psi(i,n)}$ is an open cover of X.

Define $g : \mathbb{N} \to \mathbb{N}$. For $m \in \mathbb{N}$, let $g(m) := 1 + \max\{k \in \mathbb{N} \mid (m, k)^\uparrow \in \mathcal{F}_m\}$. The definition is good since $\mathcal{F}_m \subseteq \mathcal{U}_m = \{(m, k)^\uparrow \mid k \in \mathbb{N}\}$ and finite. We claim that g witnesses that X is not-dominating. We pick $f \in X$ and show that $\chi_{f,g} := \{m \in \mathbb{N} \mid g(m) \not\leq f(m)\}$ is infinite. We do this by introducing some $h \in \mathbb{N}^\mathbb{N}$ with the property that $\{\psi\big(i, h(i)\big) \mid i \in \mathbb{N}\} \subseteq \chi_{f,g}$.

Fix $i \in \mathbb{N}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\psi(i,n)}$ is an open cover of X, there exists some $n \in \mathbb{N}$ such that $f \in \mathcal{F}_{\psi(i,n)}$, so let $h(i) := n$ for such an n. End of definition. It follows that $f \in \mathcal{F}_{\psi(i,h(i))}$ for all $i \in \mathbb{N}$, and hence $f(\psi(i, h(i))) \leq g(\psi(i, h(i)))-1$. In particular, $\forall i \in \mathbb{N}\big(\psi(i, h(i)) \in \chi_{f,h}\big)$. □

Theorem 4.12 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 1.31, we assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N}\rangle$. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. Now, for all $n, m \in \mathbb{N}$, let $V_n^m := U_n^m \setminus \bigcup_{k<m} U_n^k$.

By the hypothesis on $\mathcal{B}$, V_n^m are open for all $n, m \in \mathbb{N}$.

It follows that we may assume for all $n \in \mathbb{N}$ that members of $\mathcal{U}_n$ are mutually-disjoint, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^\mathbb{N}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^\mathbb{N}$ be the map $x \mapsto f_x$.

To see that ψ continuous, fix some some $n \in \mathbb{N}$ and $\sigma : \{1, .., n\} \to \mathbb{N}$. We shall show that $\psi^{-1}[\sigma^\uparrow]$ is open. Indeed, by definition, $\psi^{-1}[\sigma^\uparrow] = \bigcap_{k=1}^n U_k^{\sigma(k)}$ which is a finite intersection of open sets, thus, open.

Let $g \in \mathbb{N}^\mathbb{N}$ be a witness to the fact that $\psi[X]$ is non-dominating. For all $n \in \mathbb{N}$, put $\mathcal{F}_n := \{U_n^1, .., U_n^{g(n)}\}$. We claim that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X. To see this, fix $x \in X$. By definition of g, there must exist some $n \in \mathbb{N}$ with $g(n) \nleq f_x(n)$, that is, there exists some $k < g(n)$ such that $x \in U_n^k$, and clearly $U_n^k \in \mathcal{F}_n$. It follows that $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{F}_n$. □

Corollary 4.13. *If $X \in [\mathbb{R}]^{<\mathfrak{d}}$, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. By ($\Leftarrow$) of Theorem 4.10. □

We now get a result stronger than 3.19, but is only limited to subspaces of the real line.

Corollary 4.14. *Suppose $Y \subseteq X \subseteq \mathbb{R}$ are such that:*

- *$Y \models S_{fin}(\mathcal{O}, \mathcal{O})$;*
- *X is $\mathfrak{d}$-concentrated at Y.*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 3.17 and the preceding Corollary. □

Corollary 4.15. *If $X \subseteq \mathbb{R}$ is $\mathfrak{d}$-concentrated at some $Y \in [\mathbb{R}]^{<\mathfrak{d}}$, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Theorem 4.16. *Suppose $X \subseteq \mathbb{R}$ is $\mathfrak{c}$-concentrated at some countable $D \subseteq X$, then X does not contain a perfect subset.*

Proof. Suppose not, and let X be a witness to that. By Lemma 3.26, X contains an homeomorphic copy of $\{0, 1\}^\mathbb{N}$, thus it suffices to prove the following. □

Lemma 4.17. *$\{0, 1\}^\omega$ is not $\mathfrak{c}$-concentrated at any of its countable subsets.*

Proof. Let $D = \{f_n \mid n \in \omega\}$ be a countable subset of $\{0, 1\}^\omega$.

For $n \in \omega$, $U_n := (f_n \restriction \{2n, 2n + 1\})^\uparrow$ is an open set containing f_n. It follows that $D \subseteq U$ where $U := \bigcup_{n\in\omega} U_n$. We are left with showing that $\{0, 1\}^\omega \setminus U$ is of cardinality $\mathfrak{c}$.

Indeed, for each $a:\omega\to\{0,1\}$, let $f_a:\omega\to\{0,1\}$ be the function satisfying for all $n\in\omega$:

$$f_a(2n+a(n)) = f_n(2n+a(n)) \quad \text{and:}$$
$$f_a(2n+1-a(n)) = 1-f_n(2n+1-a(n)).$$

It follows that $\{k\in\omega \mid f_n(k)=f_a(k)\}$ and $\{k\in\omega\mid f_n(k)\neq f_a(k)\}$ are both non-empty for all $n\in\omega$. More importantly, $a\mapsto f_a$ is injective. Thus, $\{f_a \mid a\in {}^{\omega}\{0,1\}\}$ is a subset of $\{0,1\}^{\omega}$ of cardinality $\mathfrak{c}$ and disjoint from the open set U containing D. $\square$

Corollary 4.18. *If $X\subseteq\mathbb{R}$ is uncountable and $\mathfrak{d}$-concentrated at some $Y\in[\mathbb{R}]^{\aleph_0}$, then X is a counter-example to Menger's conjecture 1.30.*

Proof. By Corollary 4.15, Lemma 3.23 and Theorem 4.16. $\square$

Corollary 4.19. *For all $X\subseteq\mathbb{R}$, if $\aleph_0<|X|<\mathfrak{d}$, then X is a counter-example to Menger's conjecture 1.30.*

In particular, if $\mathfrak{d}>\aleph_1$, then there exists a counter-example to the conjecture.

Theorem 4.20 (Fremlin-Miller)**.** *Menger's conjecture 1.30 is false.*

Proof by Bartoszyński-Tsaban. Let $D\subseteq\mathbb{N}^{\mathbb{N}}$ be a $\mathfrak{d}$-scale (see Lemma 1.12) and $\psi:\mathbb{N}^{\mathbb{N}}\leftrightarrow[0,1]\setminus\mathbb{Q}$ be an homemorphism (see Theorem 2.29). Consider $M:=\psi[D]\cup(\mathbb{Q}\cap[0,1])$.

We shall show that M is $\mathfrak{d}$-concentrated at $\mathbb{Q}\cap[0,1]$. Suppose that $U\subseteq\mathbb{R}$ is open and $U\supset(\mathbb{Q}\cap[0,1])$. It follows that:

$$|M\setminus U| = |\psi[D]\cap([0,1]\setminus U)| = |D\cap K|,$$

where $K:=\psi^{-1}([0,1]\setminus U)$.

Since $([0,1]\setminus U)$ is a closed subset of the bounded interval $[0,1]$, it is compact, and hence K is compact. Applying Lemma 4.3 on K, we find some $g\in\mathbb{N}^{\mathbb{N}}$ such that $K\subseteq\underline{\{g\}}$. Finally, since D is a $\mathfrak{d}$-scale we conclude that $|M\setminus U|=|D\cap K|\leq|D\cap\underline{\{g\}}|<\mathfrak{d}$. $\square$

Similarly, If $B\subseteq\mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-scale, then $H:=\psi[B]\cup(\mathbb{Q}\cap[0,1])$ is $\mathfrak{b}$-concentrated at $\mathbb{Q}\cap[0,1]$, thus, $H\subseteq\mathbb{R}$ is another counter-example to Menger's conjecture.

We next give a little background on connectedness.

Definition 4.21. A space X is *disconnected* iff there are disjoint open sets H,K such that $X=H\cup K$. When no such disconnection exists, X is *connected.*

A space X is *totally disconnected* iff for every $x\in X$ the only connected set containing x is $\{x\}$.

Note that we can replace "open" in the definition by "closed". It is apparent, then, that X is connected iff there are no clopen (open-closed) subsets of X but X itself and $\emptyset$.

The Cantor set, the rationals and the irrationals, are all totally disconnected spaces.

Definition 4.22. A space X is *0-dimensional* iff X has a base consisting of only clopen sets.

Equivalently, X is 0-dimensional iff for each $x \in X$ and a closed set $A \subset X$ not containing x, there is a clopen set containing x and disjoint from A. By this, the following is immediate.

Proposition 4.23. *Every 0-dimensional T_1 space* [13] *is totally disconnected.*

Lemma 4.24. *If X is a compact, totally disconnected Hausdorff space, then whenever $x \neq y$ in X, there is a clopen set in X containing x but not y.*

Definition 4.25. A space $\langle X, O \rangle$ is locally compact iff whenever $x \notin A$ where A is closed, there is an open set with a compact closure disjoint from A.

Observation 4.26. *If $\langle X, O \rangle$ is a compact topological space and $Y \subseteq X$ is a closed subspace, then Y is compact.*

Corollary 4.27. *Locally compact is an hereditary property.*

Theorem 4.28. *A locally compact, Hausdorff space is 0-dimensional iff it is totally disconnected.*

Proof. It suffices to that prove a locally compact, totally disconnected Hausdorff space is 0-dimensional.

Assume A is a closed set in X, where $x \notin A$. Let U be an open set with compact closure such that $x \in U \subseteq \overline{U} \subseteq A^c$. For each $p \in \overline{U} \setminus U$, let V_p be a clopen subset of $\overline{U}$ containing x but not p. The sets $X \setminus V_p$ form an open cover of $\overline{U} \setminus U$ so a finite subcover exists , say corresponding to the points $p_1, ..., p_n$. Let $V := V_{p_1} \cap \cdots \cap V_{p_n}$. Then V is clopen in $\overline{U}$ containing x and disjoint from $\overline{U} \setminus U$. But then $V \subset U$ and hence is a clopen set in X containing x and disjoint from A. We conclude that X is 0-dimensional. □

[13] X is T_1 iff for every $x \neq y$ in X there is an open set containing x but not y.